On anelliptic approximations for qP velocities in TI and orthorhombic media

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ABSTRACT

Anelliptic approximations for phase and group velocities of qP waves in transversely isotropic (TI) media have been widely applied in various seismic data processing and imaging tasks. We revisit previously proposed approximations and suggest two improvements. The first improvement involves finding an empirical connection between anelliptic parameters along different fitting axes based on laboratory measurements of anisotropy of rock samples of different types. The relationship between anelliptic parameters observed is strongly linear suggesting a novel set of anisotropic parameters suitable for the study of qP-wave signatures. The second improvement involves suggesting a new functional form for the anelliptic parameter term to achieve better fitting along the horizontal axis. These modifications lead to improved three-parameter and four-parameter approximations for phase and group velocities of qP waves in TI media. In a number of model comparisons, the new three-parameter approximations appear to be more accurate than previous approximations with the same number of parameters. These modifications also serve as a foundation for an extension to orthorhombic media where qP velocities involve nine independent elastic parameters. However, as shown by previous researchers, qP wave propagation in orthorhombic media can be adequately approximated using just six combinations of those nine parameters. We propose novel six-parameter approximations for phase and group velocities for qP waves in orthorhombic media. The proposed orthorhombic phase-velocity approximation provides a more accurate alternative to previously known approximations and can find applications in full-wave modeling, imaging, and inversion. The proposed group-velocity approximation is also highly accurate and can find applications in ray tracing and velocity analysis.

INTRODUCTION

Anellipticity is a well-known characteristic of elastic wave propagation in anisotropic media. The simplest, yet practically important case of anellipticity, occurs in transversely isotropic media (Grechka, 2009; Tsvankin, 2012; Thomsen, 2014). In recent years, it has been recognized that transverse isotropy may not be sufficient to characterize the actual media encountered in many regions of the world and as a result, orthorhombic anisotropy has become a significant topic of interest (e.g. Tsvankin, 1997, TCCS-9).
qP velocities approximations

2012; Bakulin et al., 2000; Xu et al., 2005; Vasconcelos and Tsvankin, 2006; Grechka, 2009; Fowler et al., 2014; Thomsen, 2014). One important example of an orthorhombic medium is a sedimentary basin exhibiting parallel vertical cracks embedded in a background medium with vertical transverse isotropy (Schoenberg and Helbig, 1997; Tsvankin, 1997, 2012; Grechka, 2009). In such media, three-dimensional anellipticity remains an important characteristic of elastic wave propagation. Tsvankin (1997, 2012) pointed out that the elastic wave propagation in TI media resembles the elastic wave propagation in the symmetry plane of orthorhombic media. This observation enables an accurate description of orthorhombic anellipticity using only a limited number of parameters by extending the approach used to approximate anellipticity in TI media.

The exact expressions for qP phase and group velocities in TI media involve four independent parameters (Gassmann, 1964; Berryman, 1979). Alkhalifah and Tsvankin (1995) and Alkhalifah (1998) showed that three combinations of those four parameters are sufficient to describe qP wave propagation with high accuracy. Although the exact expression of phase velocity in terms of phase angle is known, the exact expression for group velocity in terms of group angle appears too complicated for practical use. Therefore, accurate approximations involving a small number of independent parameters are needed. In orthorhombic media, the exact expression for qP phase velocity can be derived as a solution of a cubic equation and involves nine parameters. However, only six combinations of those nine parameters are sufficient to accurately describe qP wave propagation (Tsvankin, 1997, 2012). The exact expression of qP group velocity in orthorhombic media can be derived from phase velocity expressions, but this expression is again cumbersome and can only be expressed in terms of the phase angle instead of group angle. Therefore, this expression is not always convenient for practical applications such as ray tracing and moveout correction, where the expression in terms of the group angle (seismic ray direction) is often preferred.

Many approximations have been proposed previously for both phase and group velocities in TI media (e.g. Dellinger et al., 1993; Alkhalifah and Tsvankin, 1995; Tsvankin, 1996; Alkhalifah, 1998, 2000a,b; Schoenberg and de Hoop, 2000; Stopin, 2001; Zhang and Uren, 2001; Daley et al., 2004; Fomel, 2004; Ursin and Stovas, 2006; Fomel and Stovas, 2010; Stovas, 2010; Farra and Pšenčík, 2013). Fowler (2003) presented a comprehensive comparative review of many of these approximations. Accuracy comparison of several group-velocity approximations (in terms of moveout approximations) was also presented by Aleixo and Schleicher (2010) and Golikov and Stovas (2012). Among these different approaches, Fomel (2004) proposed an extension of the Muir-Dellinger approach (Muir and Dellinger, 1985; Dellinger et al., 1993) using the shifted-hyperbola functional form. The resultant three-parameter approximation for phase velocity is identical to the acoustic approximation of Alkhalifah (1998, 2000a) and the empirical approximation of Stopin (2001). The corresponding three-parameter approximation for group velocity was new at the time and proved to be exceptionally accurate in comparison with other known approximations.

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In the first part of this study, we revisit the anelliptic approximations by Fomel (2004) and further improve their accuracy by using an empirical relationship between the vertical and horizontal anelliptic parameters extracted from many laboratory measurements of stiffness tensor coefficients. We also modify the functional form of the approximations to improve their behavior at large angles.

Many studies of elastic wave propagation and velocity approximations in orthorhombic media have been reported in the literature, and several alternative six-parameter approximations for qP phase velocity have been proposed (Tsvankin, 1997; Alkhalifah, 2003; Grechka, 2009; Song and Alkhalifah, 2013; Hao and Stovas, 2014). Several group-velocity approximations for orthorhombic media have been proposed in the form of moveout approximations (Xu et al., 2005; Vasconcelos and Tsvankin, 2006). Using the fact that the elastic wave propagation in each of the three symmetry planes of orthorhombic media is controlled by the same Christoffel equation as in the case of TI media (Tsvankin, 1997, 2012), we develop novel approximations for orthorhombic qP velocities by starting from our approximations in TI media. We extend our anelliptic TI approximations to a 3D form suitable for approximation of phase and group velocities of qP waves in orthorhombic media. Using a set of test models, we check the accuracy of the proposed approximations and verify that they provide more accurate alternatives to the previously known approximations. In some of the models, the improvement in accuracy is dramatic and reaches a factor of ten. The proposed approximations can readily be used in seismic data processing and imaging applications. We show examples of applying the proposed phase-velocity approximations for TI and orthorhombic media in wave extrapolation experiments.

**CHOICE OF ANISOTROPIC PARAMETERS**

Previous researchers have shown that in TI and orthorhombic media, only three and six combinations of stiffness coefficients respectively are sufficient to describe qP-wave propagation. The most widely used parameters are Thomsen’s parameters (Thomsen, 1986) in TI media and their extension to orthorhombic media (Tsvankin, 1997). Another notable parameterization scheme involves the \( \eta \) parameter (Alkhalifah and Tsvankin, 1995) and is commonly used when the so-called acoustic approximation is assumed (Alkhalifah, 1998, 2000a,b, 2003). In this study, following the work of Muir and Dellinger (1985) and Dellinger et al. (1993), we adopt the set of anisotropic parameters, referred to as Muir-Dellinger parameters, which represent different combinations of elastic moduli.

In our notation, the Muir-Dellinger parameters include \( w_1, w_3, q_1, \) and \( q_3 \), where \( w_1 \) denotes the horizontal velocity squared and \( w_3 \) denotes the vertical velocity squared, and \( q_1 \) and \( q_3 \) are anelliptic parameters derived from fitting the phase velocity curvatures along the horizontal axis and vertical axis. In terms of density-normalized
stiffness tensor coefficients in Voigt notation, \( w_1 = c_{11}, \ w_3 = c_{33} \), and
\[
q_1 = \frac{c_{55}(c_{11} - c_{55}) + (c_{55} + c_{13})^2}{c_{33}(c_{11} - c_{55})}, \quad (1)
\]
\[
q_3 = \frac{c_{55}(c_{33} - c_{55}) + (c_{55} + c_{13})^2}{c_{11}(c_{33} - c_{55})}, \quad (2)
\]

Equations 1 and 2 were derived previously by Muir and Dellinger (1985). Thomsen’s parameters and the parameter \( \eta \) used by Alkhalifah and Tsvankin (1995) are related to \( q_1 \) and \( q_3 \) by the conversion shown in Table 1. Table 2 shows a comparison between parameterization schemes for anisotropic parameters in TI media. Uncertainty analysis for different choices of anisotropic parameters in TI media is discussed in Appendix A.

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### Table 1: Conversion table between anelliptic parameters, Thomsen’s parameters (Thomsen, 1986) and the time-processing parameter \( \eta \) (Alkhalifah and Tsvankin, 1995).

<table>
<thead>
<tr>
<th>Anelliptic</th>
<th>Thomsen</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_1 ) = 1 + \frac{2(R-1)(\delta-\epsilon)}{R-(1+2\epsilon)}</td>
<td>( \delta = \frac{(q_3-q_1) + R(q_1q_3-2q_3+1)}{2(q_1-q_3+R(q_3-1))} )</td>
</tr>
<tr>
<td>( q_3 ) = \frac{1+2\delta}{1+2\epsilon} = \frac{1}{1+2\eta}</td>
<td>( \epsilon = \frac{(q_1-q_3)(R-1)}{2(q_1-q_3+R(q_3-1))} )</td>
</tr>
</tbody>
</table>

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Table 2: Comparison of four-parameter parameterization schemes for qP-wave anisotropic parameters.

<table>
<thead>
<tr>
<th>Schemes</th>
<th>Parameters</th>
<th>Elliptical Ani.</th>
<th>Acoustic Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thomsen (1986)</td>
<td>( V_{P0}, V_{S0}, \epsilon, ) and ( \delta )</td>
<td>( \epsilon = \delta )</td>
<td>( V_{S0} = 0 )</td>
</tr>
<tr>
<td>Alkhalifah (1998)</td>
<td>( V_{P0}, V_{S0}, V_{nmo} ) and ( \eta )</td>
<td>( \eta = 0 )</td>
<td>( V_{S0} = 0 )</td>
</tr>
<tr>
<td>Muir and Dellinger (1985), Fomel (2004), and proposed</td>
<td>( w_1, w_3, q_1 ) and ( q_3 )</td>
<td>( q_1 = q_3 = 1 )</td>
<td>( q_1 = q_3 )</td>
</tr>
</tbody>
</table>

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To study correlations between anisotropic parameters, we compiled laboratory measurements of stiffness tensor elements or other equivalent measurements of anisotropy (Jones and Wang, 1981; Thomsen, 1986; Schoenberg and Helbig, 1997; Vernik and Liu, 1997; Wang, 2002). Computing \( q_1 \) and \( q_3 \) according to equations 1 and 2, we discover an empirical relationship between anelliptic parameters shown in Figure 1.
The results suggest that $q_1$ and $q_3$ exhibit a linear relationship ($q_1 = aq_3 + b$) that appears to depend solely on lithology, regardless of the geographical location of the samples. We can also observe that the proportionality constant $a$ is close to 1 for nearly isotropic rocks (carbonates and sandstones) but deviates from 1 for highly anisotropic rocks (shales). We assume that each relationship between $q_1$ and $q_3$ given in Figure 1 is valid for that type of media and therefore, the proportionality constant $a$ and the intercept $b$ are not new independent parameters.

In consideration of a linear relationship ($q_1 = aq_3 + b$) above, the ratio of vertical qP- and qSV-wave velocities squared $R = V_{S0}^2 / V_{P0}^2$ can be expressed as,

$$R = \frac{(1 + 2\epsilon)[(1 + 2\delta)(a - 1) + b(1 + 2\epsilon)]}{a(1 + 2\delta) + (1 + 2\epsilon)(b - 1 - 2(\delta - \epsilon))}. \quad (3)$$

Therefore, if $a = 1$ and $b = 0$, $R = 0$. This result agrees with that of Fomel (2004), who previously showed that if $w_1 \neq w_3$, $q_1 \neq 1$, and $q_3 \neq 1$, setting $q_1 = q_3$ is equivalent to the assumption used in the acoustic approximation (Alkhalifah, 1998). In other words,

$$\lim_{R \to 0} q_1 = q_3. \quad (4)$$

A similar condition is discussed by Fowler (2003).

Because the in-plane qP-wave propagation in orthorhombic media behaves identically to the case of TI media (Tsvankin, 2012), we can extend the Muir-Dellinger parameters from 2D to 3D appropriately for studies of orthorhombic media. The full set of parameters in 3D includes $w_1$, $w_2$, $w_3$, $q_{21}$, $q_{31}$, $q_{12}$, $q_{32}$, $q_{13}$, and $q_{23}$, where $w_i$ denotes the velocity squared in the $n_i$ direction and $q_{ij}$ denotes the anelliptic parameters derived from fitting the phase velocity curvatures along the $n_i$ axis in the symmetry plane defined by the $n_j$ axis (Figure 3b). For example, in plane $[n_1, n_3]$, we consider $q_{12}$ and $q_{32}$ because we can find $q$ either by fitting along $n_1$ or $n_3$ axis with $n_2$ as the axis defining the symmetry plane. The expressions for the anelliptic parameters are as follows:

$$q_{21} = \frac{c_{44}(c_{22} - c_{44}) + (c_{44} + c_{23})^2}{c_{33}(c_{22} - c_{44})}, \quad (5)$$

$$q_{31} = \frac{c_{44}(c_{33} - c_{44}) + (c_{44} + c_{23})^2}{c_{22}(c_{33} - c_{44})}, \quad (6)$$

$$q_{13} = \frac{c_{66}(c_{11} - c_{66}) + (c_{66} + c_{12})^2}{c_{22}(c_{11} - c_{66})}, \quad (7)$$

$$q_{23} = \frac{c_{66}(c_{22} - c_{66}) + (c_{66} + c_{12})^2}{c_{11}(c_{22} - c_{66})}. \quad (8)$$

Expressions for $q_{12}$ and $q_{32}$ are equivalent to expressions for $q_1$ and $q_3$ in equations 1 and 2. For subscript convention on Muir-Dellinger parameters and associating expressions in 3D, we reserve $i$, $j$, and $k$ specifically to indicate the fitting location in each symmetry plane. The summation convention for repeating indices is not assumed.
Figure 1: Relationship between $q_1$ and $q_3$ in different lithology. Data are obtained from various publications on laboratory measurements. Dashed line indicates graph of $q_1 = q_3$ in each case.
Alternatively, we adopt the notation that, for each combination of $i$, $j$, and $k$, the first digit indicates the index of the fitting axis and the second digit indicates the index of the axis defining the symmetry plane. Therefore, in our notation, $i$, $j$, and $k$ are integers between 1 and 3 and in each expression, they must be different from one another. This set of $q_{ij}$ parameters may also be considered as elements of a 3×3 matrix with the diagonal elements of this matrix absent. Table 3 shows a comparison between parameterization schemes for anisotropic parameters in orthorhombic media.

<table>
<thead>
<tr>
<th>Schemes</th>
<th>Parameters</th>
<th>Acoustic Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tsvankin (1997)</td>
<td>$V_{P0}$, $V_{S0}$, $\delta^{(3)}$, $\epsilon^{(1)}$, $\delta^{(1)}$, $\gamma^{(1)}$, $\epsilon^{(2)}$, $\delta^{(2)}$, $\gamma^{(2)}$</td>
<td>-</td>
</tr>
<tr>
<td>Alkhalifah (2003)</td>
<td>$V_v$, $V_{S1}$, $V_{S2}$, $V_{S3}$, $V_{nmo1}$, $V_{nmo2}$, $\eta_1$, $\eta_2$, $\delta$</td>
<td>$V_{S1} = 0$, $V_{S2} = 0$, $V_{S3} = 0$</td>
</tr>
<tr>
<td>Proposed</td>
<td>$w_1$, $w_2$, $w_3$, $q_{12}$, $q_{21}$, $q_{13}$, $q_{31}$, $q_{23}$, $q_{32}$</td>
<td>$q_{21} = q_{31}$, $q_{12} = q_{32}$, $q_{13} = q_{23}$</td>
</tr>
</tbody>
</table>

Table 3: Comparison of parameterization schemes for qP-wave anisotropic parameters and the assumption for acoustic approximation in orthorhombic media.

On the basis of the Muir-Dellinger parameters in TI and orthorhombic media introduced in this section, we propose novel phase- and group-velocity approximations for qP waves in the subsequent sections. We first suggest a symmetric extension of the velocity approximations in TI media by Fomel (2004) and also extend the expressions to the orthorhombic case. We then utilize the relationships from Figure 1 to reduce the number of independent parameters in the proposed approximations to three and six in TI and orthorhombic media respectively so that they require the similar number of parameters as the other previously suggested approximations.

**TRANSVERSELY ISOTROPIC MEDIA**

**Exact Expression**

The phase velocity of qP waves in TI media has the following well-known explicit expression (Gassmann, 1964; Berryman, 1979):

$$v_{\text{phase}}^2 = \frac{1}{2}\left[(c_{11}+c_{55})n_1^2+(c_{33}+c_{55})n_3^2\right] + \frac{1}{2}\sqrt{\left[(c_{11}-c_{55})n_1^2-(c_{33}-c_{55})n_3^2\right]^2 + 4(c_{13}+c_{55})^2n_1^2n_3^2},$$  

(9)
where $c_{ij}$ are density-normalized stiffness tensor coefficients in Voigt notation, $n_1 = \sin \theta$, $n_3 = \cos \theta$, and $\theta$ is the phase angle (measured from the vertical axis). Group velocity can be determined from phase velocity using the general expression (Červený, 2001)

$$v_{\text{group}} = v_{\text{phase}} n + (I - nn^T) \nabla_n v_{\text{phase}},$$

(10)

where $I$ denotes the identity matrix, $n = \{n_1, n_3\}$ is the phase direction vector, and $\nabla_n v_{\text{phase}} = \{\frac{\partial v_{\text{phase}}}{\partial n_1}, \frac{\partial v_{\text{phase}}}{\partial n_3}\}^T$ is the gradient of $v_{\text{phase}}$ with respect to $n$. Using Muir-Dellinger parameters, the exact phase velocity for qP waves (equation 9) can be expressed as

$$v_{\text{phase}}^2 = \frac{1}{2} [w_1 n_1^2 + w_3 n_3^2 + w_{13}] + \frac{1}{2} \sqrt{f},$$

(11)

where

$$f = [w_1 n_1^2 + w_3 n_3^2 - w_{13}]^2 + \frac{4(q_1 - 1)(q_3 - 1)(w_3 - w_1)w_{13}n_1^2n_3^2}{(q_1 - q_3)},$$

$$w_{13} = \frac{(q_1 - q_3)w_1w_3}{(q_1 - 1)w_3 - (q_3 - 1)w_1}.$$

Muir and Dellinger Approximations

Similar to the derivations by Fomel (2004), the Muir-Dellinger approximations (Muir and Dellinger, 1985; Dellinger et al., 1993) serve as the starting point of our derivation. The Muir-Dellinger phase-velocity approximation is of the following form:

$$v_{\text{phase}}^2(n_1, n_3) \approx e(n_1, n_3) + \frac{(q - 1)w_1w_3n_1^2n_3^2}{e(n_1, n_3)},$$

(12)

where $q$ is the anelliptic parameter ($q = 1$ in case of elliptical anisotropy), $w_1 = c_{11}$ denotes the horizontal ($n_1$) velocity squared, $w_3 = c_{33}$ denotes the vertical ($n_3$) velocity squared, and $e(n_1, n_3)$ describes the elliptical part of the velocity and is defined by

$$e(n_1, n_3) = w_1 n_1^2 + w_3 n_3^2.$$

The group-velocity approximation takes a similar form, but with symmetric changes in the coefficients and variables,

$$\frac{1}{v_{\text{group}}^2(N_1, N_3)} \approx E(N_1, N_3) + \frac{(Q - 1)W_1W_3N_1^2N_3^2}{E(N_1, N_3)},$$

(14)

where $N_1 = \sin \Theta$, $N_3 = \cos \Theta$, $\Theta$ is group angle (from vertical), $W_1 = 1/w_1$ denotes the horizontal slowness squared, $W_3 = 1/w_3$ denotes the vertical slowness squared, $Q = 1/q$, and $E(N_1, N_3)$ describes the elliptical part of the slowness and is defined by

$$E(N_1, N_3) = W_1 N_1^2 + W_3 N_3^2.$$

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As suggested by Muir and Dellinger (1985), the $q$ parameter can be found by fitting the phase-velocity curvature around either the vertical axis ($\theta = 0$) or the horizontal axis ($\theta = \pi/2$). The explicit expressions of $q$ fitting in those two cases are given in equations 1 and 2. If we define $Q$ in equation 14 by fitting the group velocity curvature around either $\Theta = 0$ or $\Theta = \pi/2$, we find that

$$Q_i = 1/q_i.$$  

(16)

Extending this idea, Dellinger et al. (1993) proposed four-parameter approximations for phase and group velocities using both $q_1$ and $q_3$.

**Previous Approximations**

To obtain more accurate approximations, Fomel (2004) suggested applying the shifted-hyperbola functional form, which introduces shift parameters $s$ (for phase velocity) and $S$ (for group velocity) into the approximations using the following functional form:

$$v_{\text{phase}}^2 \approx e(n_1, n_3)(1 - s) + s\sqrt{e^2(n_1, n_3) + \frac{2(q - 1)w_1w_3n_1^2n_3^2}{s}},$$  

(17)

$$\frac{1}{v_{\text{group}}^2} \approx E(N_1, N_3)(1 - S) + S\sqrt{E^2(N_1, N_3) + \frac{2(Q - 1)W_1W_3N_1^2N_3^2}{S}}.$$  

(18)

Parameters $s$ and $S$ can be found by fitting the fourth derivatives $d^4v_{\text{phase}}/d\theta^4$ and $d^4v_{\text{group}}/d\Theta^4$ to the exact phase and group velocities, respectively. The results derived by Fomel (2004) for the vertical fitting ($\theta = 0$ or $\Theta = 0$) are

$$s = \frac{(w_1 - w_3)(q_3 - 1)(q_1 - 1)}{2[w_1(1 - q_1 - q_3(1 - q_3)) - w_3((q_1 - 1)^2 + q_1(q_3 - q_1))]};$$  

(19)

$$S = \frac{(W_3 - W_1)(Q_3 - 1)(Q_1 - 1)}{2[W_1(Q_1 - Q_3^2 + Q_3^2 - 1) + W_3(Q_1(Q_3^2 - Q_3 - 1) + 1)].}$$  

(20)

The introduction of parameters $s$ and $S$ leads to an increase in the number of parameters from three to four. To reduce this number back to three, Fomel (2004) suggested setting $q_1 = q_3$, or equivalently, $Q_1 = Q_3$, which results in $s = 1/2$ and $S = 1/2(1 + Q_3)$. Note that if we use equation 2 and set $q_1 = q_3$, this substitution will transform approximations 17 and 18 to the following form:

$$v_{\text{phase}}^2(\theta) \approx \frac{1}{2}e(\theta) + \frac{1}{2}\sqrt{e^2(\theta) + 4(q_3 - 1)w_1w_3\sin^2\theta \cos^2\theta},$$  

(21)

and

$$\frac{1}{v_{\text{group}}^2}(\Theta) \approx \frac{1 + 2Q_3}{2(1 + Q_3)}E(\Theta) + \frac{1}{2(1 + Q_3)}\sqrt{E^2(\Theta) + 4(Q_3^2 - 1)W_1W_3\sin^2\Theta \cos^2\Theta},$$  

(22)

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The phase-velocity approximation in equation 21 is equivalent to the acoustic approximation of Alkhalifah (1998, 2000a) and the empirical approximation of Stopin (2001), which were derived in a different way. As discussed in the previous section, taking \( q_1 = q_3 \), or, equivalently, \( Q_1 = Q_3 \), is not necessarily the optimal choice, because the values of these two parameters may depend on the material properties of the media of interest. The actual empirical relationship between \( q_1 \) and \( q_3 \) can be extracted from laboratory or in situ measurements of the stiffness tensor elements in various media (Figure 1). Furthermore, Equations 21 and 22 are derived by fitting the derivatives up to fourth-order at either the vertical or horizontal axis, whereas fitting at the other axis is only first-order. This low-order fitting may lead to a loss of accuracy at larger angles (\( \theta \) or \( \Theta \)).

**Proposed Approximations**

To derive a more symmetric form, we return to the four-parameter expressions (equations 17 and 18) and propose to modify them as follows:

\[
v_{\text{phase}}^2 \approx e(n_1, n_3)(1 - \hat{s}) + \hat{s} \sqrt{e^2(n_1, n_3) + \frac{2(\hat{q} - 1)w_1w_3n_1^2n_3^2}{\hat{s}}} ,
\]

and

\[
\frac{1}{v_{\text{group}}^2} \approx E(N_1, N_3)(1 - \hat{S}) + \hat{S} \sqrt{E^2(N_1, N_3) + \frac{2(\hat{Q} - 1)W_1W_3N_1^2N_3^2}{\hat{S}}} ,
\]

where

\[
\hat{q} = \frac{q_1w_1n_1^2 + q_3w_3n_3^2}{w_1n_1^2 + w_3n_3^2} , \quad \hat{Q} = \frac{Q_1W_1N_1^2 + Q_3W_3N_3^2}{W_1N_1^2 + W_3N_3^2} ,
\]

\[
\hat{s} = \frac{s_1w_1n_1^2 + s_3w_3n_3^2}{w_1n_1^2 + w_3n_3^2} , \quad \hat{S} = \frac{S_1W_1N_1^2 + S_3W_3N_3^2}{W_1N_1^2 + W_3N_3^2} .
\]

The modifications in equation 25 are equivalent to the second anelliptic approximations by Dellinger et al. (1993). Again, parameters \( q_3 \) and \( q_1 \) can be found by fitting the velocity profile curvatures at the vertical (\( \theta = 0 \)) and horizontal (\( \theta = \pi/2 \)) axis, respectively and are defined in equations 1 and 2. Analogously, \( s_3 \) and \( s_1 \) can be found by fitting the fourth-order derivative (\( d^4v_{\text{phase}}/d\theta^4 \)) at the same angle. A similar strategy applies to fitting parameters for the group-velocity approximation. Note that expressions for \( s_3 \) and \( S_3 \) are different from equations 19 and 20.

Following this approach, we derive the following expressions for \( s_1, s_3, S_1, \) and
S_3:

\[ \begin{align*}
  s_1 & = a_1/b_1 , \\
  a_1 & = (w_3 - w_1)(q_1 - 1)^2(q_3 - 1) , \\
  b_1 & = 2[w_3(q_1(q_1 - 2) + 3) - 2q_1q_3 + q_3^2 - 1] - w_1(q_3(q_1 - 4) + q_3 + 1) + 2q_1 - 1] , \\
  s_3 & = a_3/b_3 , \\
  a_3 & = (w_1 - w_3)(q_1 - 1)(q_3 - 1)^2 , \\
  b_3 & = 2[w_1(q_3(q_3 - 2) + 3) - 2q_1q_3 + q_3^2 - 1] - w_3(q_3(q_3 - 4) + q_1 + 1) + 2q_3 - 1] , \\
  S_1 & = A_1/B_1 , \\
  A_1 & = (W_1 - W_3)(Q_1 - 1)^2(Q_3 - 1) , \\
  B_1 & = 2[W_1(Q_3^2 + 2Q_1 + Q_1Q_3(Q_1 - 1)^2 - 1) - W_3(Q_3^2 - 2Q_1Q_3 + Q_1(Q_1 - 1)^2 + 2) - 1] , \\
  S_3 & = A_3/B_3 , \\
  A_3 & = (W_3 - W_1)(Q_1 - 1)(Q_3 - 1)^2 , \\
  B_3 & = 2[W_3(Q_1^2 + 2Q_3 + Q_1Q_3(Q_3 - 2) - 1) - W_1(Q_1^2 - 2Q_1Q_3 + Q_3(Q_3 - 1)^2 + 2) - 1] .
\end{align*} \]

Note that equations 23 and 24 introduce three more parameters generating six parameters in total, namely \( w_1, w_3, q_1, q_3, s_1, \) and \( s_3 \) for equation 23 or \( W_1, W_3, Q_1, Q_3, S_1, \) and \( S_3 \) for equation 24. However, expressing \( s_i \) and \( S_i \) in terms of \( q_i \) and \( Q_i \) in equations 27-30, we effectively reduce the dependency to four parameters. This reduction leads to four-parameter anelliptic approximations, which fit up to the fourth-order accuracy along both axes. The exact phase- and group-velocity expressions also require the total of four independent parameters. However, the advantage of the proposed approximations lies in the existence of the group-velocity expression (equation 24) with analogous functional form as the phase-velocity expression (equation 23). To reduce the number of parameters to three, we utilize the linear relationships between \( q_1 \) and \( q_3 \) given in Figure 1. The required \( Q_1 \) and \( Q_3 \) parameters for the group-velocity approximations can be found from the reciprocals of \( q_1 \) and \( q_3 \) for phase-velocity approximations, as mentioned above. Therefore, both phase- and group-velocity approximations derived on the basis of this approach require the same number of parameters.

**Moveout approximation**

The group-velocity approximation in equation 24 can be easily converted into the corresponding moveout equation using the relationship between offset \( x \), vertical
distance \((z)\), and total reflection traveltime \((t)\) given by

\[
t(x) = 2\frac{\sqrt{x/2 + z^2}}{V(\arctan(x/2z))},
\]

where \(z = t_0V(0)/2\) is the depth of the reflector, \(t_0\) is the vertical two-way reflection traveltime, and \(V(\Theta)\) is the approximated group velocity. The moveout equation corresponding to equation 24 is thus,

\[
t^2(x) = H(x)(1 - \hat{S} + \hat{S}\sqrt{H^2(x) + \frac{2(\hat{Q} - 1)t_0^2x^2}{\hat{S}Q_3V^2_{nmo}}}},
\]

where

\[
\hat{Q} = \frac{Q_1}{Q_3V^2_{nmo}x^2 + t_0^2} + \frac{Q_3t_0^2}{Q_4V^2_{nmo}x^2 + t_0^2}, \quad \hat{S} = \frac{S_1}{Q_3V^2_{nmo}x^2 + t_0^2} + \frac{S_3t_0^2}{Q_4V^2_{nmo}x^2 + t_0^2},
\]

\(V_{nmo}\) denotes the NMO-velocity (Alkhalifah and Tsvankin, 1995) and is given by

\[
V^2_{nmo} = \frac{1}{W_1Q_3} = w_1q_3 = V^2_{P0}\frac{1 + 2\xi}{1 + 2\eta} = V^2_{P0}(1 + 2\delta).
\]

\(H(x)\) denotes the hyperbolic part of the reflection traveltime squared and is given by

\[
H(x) = t_0^2 + \frac{x^2}{Q_3V^2_{nmo}}.
\]

Assuming a particular media type and using a linear relationship between \(q_1\) and \(q_3\), we reduce the number of independent moveout parameters in the similar manner. However, note that \(S_1\) (equations 29) and \(S_3\) (equation 30) also depend on \(W_1\) and \(W_3\). Therefore, to effectively reduce the number of parameters in the moveout approximation (equation 32) to three, we suggest, as an approximation, to adopt \(Q_1 = Q_3\) only for equations 29 and 30, which lead to

\[
S_1 = S_3 = \frac{1}{2(1 + Q_3)}.
\]

As a result, the moveout approximation depends on \(t_0\), \(V_{nmo}\), and \(Q_3\).

For small offsets, the Taylor expansion of equation 32 is

\[
t^2(x) \approx t_0^2 + \frac{x^2}{V^2_{nmo}} - \frac{1 - 2S_3(Q_1 + 1) + Q_3(4S_3 + Q_3 - 2)x^4}{2S_3Q_3^2t_0^2V^4_{nmo}},
\]

which reduces to the expression given by Fomel (2004) by setting \(Q_1 = Q_3\). The asymptote of this expression for unbounded offset \(x\) is given by

\[
\frac{1}{Q_3V^2_{nmo}} = \frac{1}{w_1},
\]

which is the horizontal velocity squared.
In the Muir-Dellinger notation, another nonhyperbolic moveout approximation, the generalized nonhyperbolic moveout approximation (Fomel and Stovas, 2010; Stovas, 2010) can be expressed as

\[
t^2(x) \approx t_0^2 + \frac{x^2}{V_{nmo}^2} + \frac{A x^4}{V_{nmo}^4 \left( t_0^4 + B x^2 + \sqrt{t_0^4 + 2B t_0^2 x^2 + C x^4} \right)} ,
\]

(38)

\[
A = \frac{(Q_3 - 1)^2 (Q_1 W_3 - Q_3 W_1)}{Q_3 (Q_1 - 1) (W_1 - W_3)} ,
\]

\[
B = \frac{(Q_3 - 1) [(2Q_3^2 - 1) W_1 + W_3 - 2Q_1 Q_3 W_3]}{Q_3 (Q_1 - 1) (W_1 - W_3)} ,
\]

\[
C = \frac{(Q_3 - 1)^2 (Q_1 - 1)^2 Q_3^2}{Q_1 - 1^2 Q_3^2} .
\]

If the empirical assumption of \(Q_1 = Q_3\), or equivalently acoustic approximation is used, equation 38 reduces to the moveout approximation of Fomel (2004).

**Examples**

To investigate the accuracy of the proposed approximations, we make the relative error comparison with both plots and tables using several anisotropy models based on values from laboratory rock samples. The plots in Figure 2 are generated using the stiffness tensor measurements of Greenhorn shales (Jones and Wang, 1981), which have been applied for various approximation comparisons in the past (e.g. Dellinger, 1991; Fomel, 2004; Stovas, 2010; Farra and Pšenčík, 2013). Additionally, Tables 4 and 5 show the RMS relative error results of the new approximations, in comparison with results from some of the previously suggested approximations using the normalized stiffness tensor measurements given in Table 6. The RMS error computation is based on

\[
\text{RMS error} = \sqrt{\frac{1}{90} \sum_{\psi=0}^{90} (v_{\text{exact}}(\psi) - v_{\text{approx}}(\psi))^2} ,
\]

(39)

where \(\psi\) denotes phase or group angle as appropriate. In all comparisons, we apply the relationships shown in Figure 1 to reduce the number of parameters from four to three. For each model, the best-performing approximation is denoted in red and bold. The proposed approximations appear to be the most accurate in nearly all of the cases.

**ORTHORHOMBIC MEDIA**

**Exact Expressions**

As the analog of equation 9, qP waves have the following well-known explicit exact expression for phase velocity in orthorhombic media (Schoenberg and Helbig, 1997;
Sripanich & Fomel

Table 4: RMS relative error (%) from 0-90° of phase-velocity approximations by Thomsen (1986), Alkhalifah (1998) (similar to Fomel (2004)), and of the proposed three-parameter approximation for transversally-isotropic elastic models from Table 6. Bold red highlight indicates the best-performing approximation. In all the cases, except sample 4, the proposed approximation appears to be the most accurate.

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<tr>
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<td>0.6789</td>
<td>0.1422</td>
<td>0.0978</td>
</tr>
<tr>
<td>2</td>
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<td>0.2254</td>
<td>0.0503</td>
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<td>3</td>
<td>0.4564</td>
<td>0.1399</td>
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<td>4</td>
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<td>0.0506</td>
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<td>0.0201</td>
</tr>
<tr>
<td>6</td>
<td>0.5710</td>
<td>0.1631</td>
<td>0.0149</td>
</tr>
</tbody>
</table>

Table 5: RMS relative error (%) from 0-90° of group-velocity approximations by Alkhalifah and Tsvankin (1995), Fomel (2004), Farra and Pšenčík (2013) (second-order) and of the proposed three-parameter approximation for transversally-isotropic elastic models from Table 6. Bold red highlight indicates the best-performing approximation. In all the cases, except samples 4 and 5, the proposed approximation appears to be the most accurate.

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>1</td>
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<td>0.0801</td>
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</tr>
<tr>
<td>6</td>
<td>0.4258</td>
<td>0.1541</td>
<td>0.1412</td>
<td><strong>0.0084</strong></td>
</tr>
</tbody>
</table>

Table 6: Normalized stiffness tensor coefficients (in km$^2$/s$^2$) from different TI samples: 1 is from Jones and Wang (1981), 2 and 3 are from Wang (2002), 4 and 5 are from Thomsen (1986), and 6 is from Vernik and Liu (1997).

<table>
<thead>
<tr>
<th>Shales sample</th>
<th>$c_{11}$</th>
<th>$c_{33}$</th>
<th>$c_{13}$</th>
<th>$c_{55}$</th>
<th>$V_{P0}$</th>
<th>$V_{S0}$</th>
<th>$\epsilon$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Greenhorn</td>
<td>14.47</td>
<td>9.57</td>
<td>4.51</td>
<td>2.28</td>
<td>3.094</td>
<td>1.510</td>
<td>0.256</td>
<td>-0.0505</td>
</tr>
<tr>
<td>2. Hard (brine)</td>
<td>20.89</td>
<td>13.89</td>
<td>3.048</td>
<td>5.655</td>
<td>3.727</td>
<td>2.378</td>
<td>0.252</td>
<td>0.0347</td>
</tr>
<tr>
<td>3. North Sea (brine)</td>
<td>7.292</td>
<td>5.248</td>
<td>1.578</td>
<td>1.798</td>
<td>2.291</td>
<td>1.341</td>
<td>0.195</td>
<td>-0.0139</td>
</tr>
<tr>
<td>4. Dog Creek</td>
<td>5.098</td>
<td>3.5163</td>
<td>2.4832</td>
<td>0.6823</td>
<td>1.875</td>
<td>0.826</td>
<td>0.225</td>
<td>0.0998</td>
</tr>
<tr>
<td>5. Mesaverde</td>
<td>17.653</td>
<td>14.055</td>
<td>1.3391</td>
<td>6.87</td>
<td>3.749</td>
<td>2.621</td>
<td>0.128</td>
<td>0.0781</td>
</tr>
<tr>
<td>6. North Sea (dry)</td>
<td>22.051</td>
<td>14.90</td>
<td>5.336</td>
<td>4.928</td>
<td>3.860</td>
<td>2.220</td>
<td>0.240</td>
<td>0.0199</td>
</tr>
</tbody>
</table>

TCCS-9
Figure 2: Relative error plots using Greenhorn Shale measurements. a) Phase velocity. b) Group velocity. c) Group velocity (finer scale).

\[ v_{\text{phase}}^2 = 2\sqrt{-\frac{d}{3}} \cos\left(\frac{\nu}{3}\right) - \frac{a}{3}, \]  

(40)

where

\[ \nu = \arccos \left( \frac{-q}{2\sqrt{(-d/3)^3}} \right), \]

\[ q = 2 \left( \frac{a}{3} \right)^3 - \frac{ab}{3} + c, \quad d = -\frac{a^2}{3} + b, \]

\[ a = -(G_{11} + G_{22} + G_{33}), \]

\[ b = G_{11}G_{22} + G_{11}G_{33} + G_{22}G_{33} - G_{12}^2 - G_{13}^2 - G_{23}^2, \]

\[ c = G_{11}G_{23}^2 + G_{22}G_{13}^2 + G_{33}G_{12}^2 - G_{11}G_{22}G_{33} - 2G_{12}G_{13}G_{23}, \]

and

\[ G_{11} = c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2, \]

\[ G_{22} = c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, \]

\[ G_{33} = c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2, \]

\[ G_{12} = (c_{12} + c_{66})n_1n_2, \]

\[ G_{13} = (c_{13} + c_{55})n_1n_3. \]

Here, \( c_{ij} \) are density-normalized stiffness tensor coefficients, \( n_1 = \sin \theta \cos \phi, \) \( n_2 = \sin \theta \sin \phi, \) \( n_3 = \cos \theta, \) \( \theta \) is zenith phase angle (measured from \( n_3 \)), and \( \phi \) is azimuthal phase angle (measured from \( n_1 \)) in the local orthorhombic frame of reference where the axes \( n_1, n_2, \) and \( n_3 \) are intersections of the corresponding planes of symmetry. The corresponding group-velocity expression can be determined from equation 10 extended to 3D.

**Extended Muir-Dellinger Approximations**

Considering the Muir-Dellinger approximations for qP velocities in TI media (equations 12 and 14) and the subscript convention introduce in the first section, we can naturally extend them to orthorhombic media as follows:

\[ v_{\text{phase}}^2(n_1, n_2, n_3) \approx e(n_1, n_2, n_3) + \sum_{j=1}^{3} \frac{(q_{ij} - 1)w_j n_i n_j^2}{e(n_1, n_2, n_3)} n_k^2, \]  

(41)

where \( q_{ij} \) are the anelliptic parameters (\( q_{ij} = 1 \) in case of elliptical anisotropy), \( w_i = c_{ii} \) denotes velocity squared along the \( n_i \) axis, and \( e(n_1, n_2, n_3) \) describes the ellipsoidal part of the velocity and is defined by

\[ e(n_1, n_2, n_3) = w_1n_1^2 + w_2n_2^2 + w_3n_3^2. \]  

(42)

_TCCS-9_
This extension is based on the consideration of an ellipsoid in 3D as opposed to an ellipse in 2D and the additional two anelliptic terms involving $q_{ij}$ parameters from considering a total of three symmetry planes in orthorhombic media. It is also valid to consider $q_{kj}$ instead of $q_{ij}$ because in consistent with the original Muir-Dellinger approximation (equation 12), only one anelliptic parameter is needed in each symmetry plane. According to the Muir-Dellinger approach, we can also derive the group-velocity approximation, which takes a similar form, with symmetric changes in the coefficients and variables as shown below:

$$\frac{1}{v_{\text{group}}^2(N_1, N_2, N_3)} \approx E(N_1, N_2, N_3) + \sum_{j=1}^{3} \frac{(Q_{ij} - 1)W_i W_k N_j^2 N_k^2}{E(N_1, N_2, N_3)},$$

(43)

where $N_1 = \sin \Theta \cos \Phi$, $N_2 = \sin \Theta \sin \Phi$, $N_3 = \cos \Theta$, $\Theta$ is zenith group angle (from vertical), $\Phi$ is azimuthal group angle (from $n_1$), $Q_{ij} = 1/q_{ij}$, $W_i = 1/w_i$ denotes slowness squared along the $N_i$ axis, and $E(N_1, N_2, N_3)$ describes the elliptical part of the slowness, defined by

$$E(N_1, N_2, N_3) = W_1 N_1^2 + W_2 N_2^2 + W_3 N_3^2.$$

(44)

This simple extension from 2D to 3D stems from the observation that elastic wave propagation in each symmetry plane of orthorhombic media is controlled by the same Christoffel equation as in the case of TI media (Tsvankin, 1997, 2012). Therefore, if any $n_i$ or $N_i$ is zero, the extended expressions will simply reduce to the 2D Muir-Dellinger approximations for TI media in equations 12 and 14. Note that the expression in equation 41 is equivalent to the two leading terms of the phase-velocity pseudo-acoustic approximation derived by Fowler and Lapilli (2012) and Fowler et al. (2014).

**Proposed Approximations**

In the preceding section, we suggested an improvement to the anelliptic approximations for VTI media previously proposed by Fomel (2004). Applying a similar modification to the extended Muir and Dellinger approximations (equations 41 and 43), we can write the resultant approximations in a new form suitable for approximation of velocities in orthorhombic media, as follows:

$$v_{\text{phase}}^2 \approx e(n_1, n_2, n_3)(1 - \hat{s}) + \hat{s}\sqrt{e^2(n_1, n_2, n_3) + \frac{2}{\hat{s}} \sum_{j=1}^{3} (\hat{q}_j - 1) w_i w_k n_j^2 n_k^2},$$

(45)

where

$$\hat{s} = \frac{\hat{s}_1 w_1 n_1^2 + \hat{s}_2 w_2 n_2^2 + \hat{s}_3 w_3 n_3^2}{w_1 n_1^2 + w_2 n_2^2 + w_3 n_3^2},$$

(46)
\[
\frac{1}{v_{\text{group}}^2} \approx E(N_1, N_2, N_3)(1 - \hat{S}) + \hat{S} \sqrt{E^2(N_1, N_2, N_3) + \frac{2}{\hat{S}} \sum_{j=1}^{3}(\hat{Q}_j - 1)W_i W_k N_i^2 N_k^2},
\]

where

\[
\hat{S} = \frac{\hat{S}_1 W_1 N_1^2 + \hat{S}_2 W_2 N_2^2 + \hat{S}_3 W_3 N_3^2}{W_1 N_1^2 + W_2 N_2^2 + W_3 N_3^2},
\]

\[
\hat{q}_j = \frac{q_{ij} W_i N_i^2 + q_{kj} W_k N_k^2}{w_i n_i^2 + w_k n_k^2}, \quad \hat{Q}_j = \frac{Q_{ij} W_i N_i^2 + Q_{kj} W_k N_k^2}{W_i N_i^2 + W_k N_k^2},
\]

\[
\hat{s}_j = \frac{s_{ij} W_i N_i^2 + s_{kj} W_k N_k^2}{w_i n_i^2 + w_k n_k^2}, \quad \hat{S}_j = \frac{S_{ijk} W_i N_i^2 + S_{jki} W_k N_k^2}{W_i N_i^2 + W_k N_k^2}.
\]

Figure 3b shows the locations of the fitting indices in each symmetry plane. Note that the subscript rule explained earlier still applies, and the relationship \( Q_{ij} = 1/q_{ij} \) still holds. In our notation, \( \hat{q} \) and \( \hat{Q} \) represent weighted averages of anelliptic parameters (q and Q) in a plane whereas \( \hat{s} \) and \( \hat{S} \) represent weighted averages at an axis.

Figure 3: a) Parametization rule for working coordinates. b) Locations of fitting indices \( q_{ij} \) in each symmetry plane.

Similar to the derivation in the TI case, \( q_{ij} \) and \( Q_{ij} \) can be found by fitting the curvatures to the exact velocities along orthogonal directions in each of the three planes. Likewise, \( s_{ij} \) and \( S_{ij} \) can be found by fitting the fourth-order derivative \( (d^4 v_{\text{phase}} / d\theta^4 \text{ and } d^4 v_{\text{group}} / d\Theta^4) \) at the same positions. Note that the expressions of \( q_{i2}, Q_{i2}, q_{i2}, \) and \( S_{i2} \) are similar to those in the TI case for \( q_i, Q_i, s_i, \) and \( S_i \) for \( i = 1, 3 \). As a result, we need to specify parameter expressions only for the other two.
planes. Fitting both phase- and group-velocity expressions along both axes in each symmetry plane leads to six different expressions for each of the fitting parameters. This is different for the case of extended Muir-Dellinger approximations (equations 41 and 43) where only three expressions are allowed.

Since the Christoffel equation in the three symmetry planes is similar to that in TI media, the functional forms of every parameter expression remain the same. Therefore, we can compute parameters needed by equations 45 and 47 using the formulas derived in \([n_1,n_3]\) plane for the TI case. Thus, we obtain:

\[
q_{ij} = \frac{(c_{ik} + c_{pp})^2 + c_{pp}(c_{ii} - c_{pp})}{c_{kk}(c_{ii} - c_{pp})},
\]

\[
s_{ij} = a_{ij}/b_{ij},
\]

\[
a_{ij} = (w_k - w_i)(q_{ij} - 1)^2(q_{kj} - 1),
\]

\[
b_{ij} = 2[w_k(q_{ij}(q_{ij} - 2) + 3) - 2q_{ij}q_{kj} + q_{kj}^2 - 1] - w_i(q_{kj}(q_{ij}(q_{ij} - 4) + q_{kj} + 1) + 2q_{ij} - 1),
\]

\[
S_{ij} = A_{ij}/B_{ij},
\]

\[
A_{ij} = (W_i - W_k)(Q_{ij} - 1)^2(Q_{kj} - 1),
\]

\[
B_{ij} = 2[W_i(Q_{kj}^2 + 2Q_{ij} + Q_{ij}Q_{kj}(Q_{ij}(Q_{ij} - 2) - 1) - 1)
- W_k(Q_{kj}^2 - 2Q_{ij}Q_{kj} + Q_{ij}(Q_{ij}(Q_{ij} - 1)^2 + 2) - 1)].
\]

where \(p = j + 3\). Recall that the indices \(ij \neq ji\) and therefore, there are six expressions corresponding to each formula from equations 51-53. Equations 45 and 47 amount to the nine-parameter approximations for both velocities where the required parameters include \(w_1, w_2, w_3, q_{21}, q_{31}, q_{12}, q_{32}, q_{13},\) and \(q_{23}\) for equation 45 or their reciprocals, \(W_1, W_2, W_3, Q_{21}, Q_{31}, Q_{12}, Q_{32}, Q_{13},\) and \(Q_{23}\) for equation 47. However, these nine parameters can be easily reduced to six using the linear relationships between \(q_{ij}\) and \(q_{kj}\) for different lithologies from the previous discussion on TI media (Figure 1). Similar conversion rules apply for anisotropic parameters in each symmetry plane and are summarized in Table 1. Note that the required independent parameters for the group-velocity approximations derived based on Muir-Dellinger approach can be found in a one-to-one relationship simply from the reciprocals of the required parameters for phase-velocity approximations as mentioned before. Therefore, phase- and group-velocity approximations derived on the basis of this approach require exactly the same number of parameters.

**Moveout approximation**

To convert the proposed group-velocity approximation (equation 47) to the corresponding moveout approximation, we apply again the general expression given in equation 31. Adopting the same notation rules, the moveout approximation takes
the form:

\[
\begin{align*}
F &= H_{\text{ortho}}^2(x, y) + \frac{2}{\hat{S}} \left( \frac{\hat{Q}_1 - 1}{Q_{31} V_{nmo}^2} + \frac{\hat{Q}_2 - 1}{Q_{32} V_{nmo}^2} + \frac{\hat{Q}_3 - 1}{(Q_{31} V_{nmo}^2)(Q_{32} V_{nmo}^2)} \right), \\
\hat{S} &= \frac{S_{13}}{Q_{31} V_{nmo}^2} y^2 + \frac{S_{14}}{Q_{31} V_{nmo}^2} + \hat{S}_0 \tag{55}
\end{align*}
\]

\[
\begin{align*}
\hat{Q}_1 &= \frac{Q_{31}^2 y^2 + Q_{31} t_0^2}{Q_{31} V_{nmo}^2 y^2 + t_0^2}, \\
\hat{Q}_2 &= \frac{Q_{32}^2 - 1}{Q_{32} V_{nmo}^2} x^2 + \frac{Q_{32} t_0^2}{Q_{32} V_{nmo}^2}, \\
\hat{Q}_3 &= \frac{Q_{33}^2 x^2 + Q_{34} y^2}{Q_{33} V_{nmo}^2}, \\
\hat{S}_1 &= \frac{S_{13}}{Q_{31} V_{nmo}^2} y^2 + \frac{S_{14}}{Q_{31} V_{nmo}^2} + \hat{S}_0 \tag{56}
\end{align*}
\]

\[
\begin{align*}
\hat{S}_2 &= \frac{S_{23}}{Q_{32} V_{nmo}^2} x^2 + \frac{S_{24} t_0^2}{Q_{32} V_{nmo}^2}, \\
\hat{S}_3 &= \frac{S_{33} x^2 + S_{34} y^2}{Q_{33} V_{nmo}^2} \tag{57}
\end{align*}
\]

\(x\) denotes the offset in \(N_1\) direction, \(y\) denotes the offset in \(N_2\) direction, \(V_{nmo} = \sqrt{1/W_1 Q_{32}}\) denotes the NMO-velocity in \(N_1\) direction, \(V_{nmo} = \sqrt{1/W_2 Q_{31}}\) denotes the NMO-velocity in \(N_2\) direction, and \(H_{\text{ortho}}(x, y)\) denotes the hyperboloidal part of reflection traveltime squared given below,

\[
H_{\text{ortho}}(x, y) = t_0^2 + \frac{x^2}{Q_{32} V_{nmo}^2} + \frac{y^2}{Q_{31} V_{nmo}^2}. \tag{58}
\]

We apply the same strategy to reduce the number of parameters with an approximation on \(Q_{ij}\) for \(S_{ij}\) as in equation 35.

For small offset, the Taylor expansion of equation 54 is

\[
\begin{align*}
t^2(x) &\approx t_0^2 + \frac{x^2}{V_{nmo}^2} + \frac{y^2}{V_{nmo}^2} - \\
&\quad - \frac{1 - 2S_{32}(Q_{12} + 1) + Q_{32}(4S_{32} + Q_{32} - 2)x^4}{2S_{32}Q_{32}^2 t_0^2 V_{nmo}^2} - \\
&\quad - \frac{1 - 2S_{31}(Q_{21} + 1) + Q_{31}(4S_{31} + Q_{31} - 2)y^4}{2S_{31}Q_{31}^2 t_0^2 V_{nmo}^2} - \\
&\quad - \frac{S_{31}(Q_{32} - 1)^2 - S_{32}(Q_{32} - 1)(Q_{32} + 2Q_{31} - 3) - 2S_{32}^2(Q_{32} + Q_{31} - Q_{13} - 1)x^2 y^2}{2S_{32} Q_{31} Q_{32} t_0^2 V_{nmo}^2 V_{nmo}^2} \tag{59}
\end{align*}
\]

The asymptote of this expression for unbounded offsets \(x\) and \(y\) is given by

\[
\begin{align*}
\frac{1}{Q_{32} V_{nmo}^2} &= \frac{1}{w_1} \quad \text{and} \quad \frac{1}{Q_{31} V_{nmo}^2} = \frac{1}{w_2}, \tag{60}
\end{align*}
\]

which denote the horizontal velocities squared along \(N_1\) and \(N_2\) directions respectively.
Examples

To evaluate the accuracy of the proposed approximations, we produce relative error plots and tables, using several sets of normalized stiffness tensor measurements summarized in Table 7, which can be converted to any parameterization scheme (Table 3). The error plots in Figures 4-10 are generated using the standard model (Schoenberg and Helbig, 1997) and are presented as both 3D surfaces and stereographic projections with $\theta$ (or $\Theta$) changing radially and $\phi$ (or $\Phi$) changing azimuthally. The standard model assumes a shale background with a set of parallel vertical cracks; therefore, we use the following relationship between anelliptic parameters in shales: $q_1 = 0.83734q_3 + 0.1581$ to reduce the number of parameters in the vertical $[n_1,n_3]$ and $[n_2,n_3]$ planes. The anisotropy in the horizontal plane $[n_1,n_2]$, on the other hand, corresponds to a different cause, which in this case is assumed to be vertical fractures. Because we do not know a proper relationship between anelliptic parameters for such feature, we resort to the previously used assumption of $q_{13} = q_{23}$. Tables 8 and 9 show RMS relative error results of our approximations in comparison with results from some of the previously suggested approximations, which are computed based on

$$\text{RMS error} = \sqrt{\sum_{\psi_1=0}^{90} \sum_{\psi_2=0}^{90} (v_{\text{exact}}(\psi_1,\psi_2) - v_{\text{approx}}(\psi_1,\psi_2))^2},\quad (61)$$

where $\psi_1$ and $\psi_2$ denote the zenith and azimuthal phase or group angles as appropriate. The best-performing approximation is denoted in red and bold. In all examples, the proposed approximations appear to be significantly more accurate than the other known approximations.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$c_{11}$</th>
<th>$c_{22}$</th>
<th>$c_{33}$</th>
<th>$c_{44}$</th>
<th>$c_{55}$</th>
<th>$c_{66}$</th>
<th>$c_{12}$</th>
<th>$c_{23}$</th>
<th>$c_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Standard model</td>
<td>9</td>
<td>9.84</td>
<td>5.938</td>
<td>2</td>
<td>1.6</td>
<td>2.182</td>
<td>3.6</td>
<td>2.4</td>
<td>2.25</td>
</tr>
<tr>
<td>2. Tsvankin 1</td>
<td>11.7</td>
<td>13.5</td>
<td>9</td>
<td>1.728</td>
<td>1.44</td>
<td>2.246</td>
<td>8.824</td>
<td>5.981</td>
<td>5.159</td>
</tr>
<tr>
<td>3. Tsvankin 2</td>
<td>17.1</td>
<td>13.5</td>
<td>9</td>
<td>1.728</td>
<td>1.44</td>
<td>2.246</td>
<td>9.772</td>
<td>4.580</td>
<td>7.745</td>
</tr>
<tr>
<td>4. Alkhalifah 1</td>
<td>1.452</td>
<td>2.016</td>
<td>1</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>1.089</td>
<td>0.695</td>
<td>0.599</td>
</tr>
<tr>
<td>5. Alkhalifah 2</td>
<td>1.452</td>
<td>2.016</td>
<td>1</td>
<td>0.49</td>
<td>0.36</td>
<td>0.49</td>
<td>0.608</td>
<td>0.206</td>
<td>0.378</td>
</tr>
</tbody>
</table>

Table 7: Normalized stiffness tensor coefficients (in $km^2/s^2$) from different orthorhombic samples: 1 is from Schoenberg and Helbig (1997), 2 and 3 are from Tsvankin (1997), and 4 and 5 are from Alkhalifah (2003).

APPLICATION TO WAVE EXTRAPOLATION

One possible application of the proposed phase-velocity approximations (equations 23 and 45) is seismic wave extrapolation based on the anisotropic wave equation. Fomel et al. (2013) and Sun and Fomel (2013) presented a lowrank approximation method to accomplish this task. The proposed phase-velocity approximations (both in TI
Table 8: RMS relative error (%) from 0-90° (both θ and φ) of orthorhombic phase-velocity approximations by Tsvankin (1997), Alkhalifah (2003), and of proposed six-parameter approximation. Bold red highlight indicates the best-performing approximation. In all cases, the proposed approximation appears to be the most accurate.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5787</td>
<td>0.1742</td>
<td>0.1029</td>
</tr>
<tr>
<td>2</td>
<td>0.5918</td>
<td>0.0645</td>
<td>0.0275</td>
</tr>
<tr>
<td>3</td>
<td>0.7104</td>
<td>0.0952</td>
<td>0.0637</td>
</tr>
<tr>
<td>4</td>
<td>0.8960</td>
<td>0.1382</td>
<td>0.0293</td>
</tr>
<tr>
<td>5</td>
<td>1.0736</td>
<td>0.3274</td>
<td>0.2084</td>
</tr>
</tbody>
</table>

Table 9: RMS relative error (%) from 0-90° (both Θ and Φ) of orthorhombic group-velocity approximations by Xu et al. (2005) and Vasconcelos and Tsvankin (2006), and of proposed six-parameter approximation. Bold red highlight indicates the best-performing approximation. In all cases, the proposed approximation appears to be more accurate.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Xu-Vasconcelos</th>
<th>Proposed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8985</td>
<td>0.1446</td>
</tr>
<tr>
<td>2</td>
<td>0.6066</td>
<td>0.1354</td>
</tr>
<tr>
<td>3</td>
<td>0.7966</td>
<td>0.0311</td>
</tr>
<tr>
<td>4</td>
<td>0.4907</td>
<td>0.0387</td>
</tr>
<tr>
<td>5</td>
<td>0.4588</td>
<td>0.1729</td>
</tr>
</tbody>
</table>

Figure 4: Relative error of phase-velocity approximation by Tsvankin (1997). a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.
Figure 5: Relative error of phase-velocity approximation by Alkhalifah (2003). a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.

Figure 6: Relative error of proposed six-parameter phase-velocity approximation. a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.
Figure 7: Relative error of proposed nine-parameter phase-velocity approximation. a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.

Figure 8: Relative error of group-velocity approximation by Xu et al. (2005) and Vasconcelos and Tsvankin (2006). a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.
Figure 9: Relative error of the proposed six-parameter group-velocity approximation. a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.

Figure 10: Relative error of the proposed nine-parameter group-velocity approximation. a) from azimuth 0 to 90°. b) from azimuth 0 to 360°.
and orthorhombic media) are converted to their corresponding dispersion relations involving frequency and wavenumber and incorporated into the wave extrapolator formulated in the Fourier domain. An example of wave extrapolation in the complex BP 2007 TTI model is shown in Figure 11. The same portion of the model was investigated by Fomel et al. (2013) and Sun and Fomel (2013). For simplicity, we take the shear-wave velocity \( V_S^0 = \sqrt{c_{55}} \) to be \( V_P^0/2 \). The results are shown in Figure 11 and demonstrate noticeably smaller phase errors obtained from the proposed approximation.

![Figure 11](image)

Figure 11: Multiple snapshots and errors of the wavefield extrapolation results for the BP TTI 2007 model. a) Wavefield extrapolation using exact phase velocity. b) Wavefield extrapolation using proposed phase-velocity approximation (23). c) Absolute error in \( n_1 - n_3 \) plane of the acoustic approximation. d) Absolute error in \( n_1 - n_3 \) plane of the proposed approximation (six-parameter).

An example in a heterogeneous tilted orthorhombic medium is shown in Figures 12 and 13 using the parameters of the tilted orthorhombic model from Song and Alkhalifah (2013). In this model, the anisotropic parameters in the notation of Alkhalifah (2003) are specified in the range, \( v_1 = 1500 : 3088 \) m/s, \( v_2 = 1500 : 3686 \) m/s, \( v_v = 1500 : 3474 \) m/s, \( \eta_1 = 0.3, \eta_2 = 0.1 \), and \( \gamma = 1.03 \). The exact formulas for \( v_1, v_2, \) and \( v_v \) are

\[
\begin{align*}
v_1 &= 1500 + 40x^2 + 30(y - 1.5)^2 + 30(z - 1)^2, \\
v_2 &= 1500 + 60x^2 + 40(y - 1.5)^2 + 40(z - 1)^2, \\
v_v &= 1500 + 50x^2 + 35(y - 1.5)^2 + 40(z - 1)^2,
\end{align*}
\]

where \( x, y, \) and \( z \) are components in the model. These values of parameters correspond to \( q_{21} = 0.857 : 0.879, q_{31} = 0.833, q_{12} = 0.670 : 0.727, q_{32} = 0.625, \)

TCCS-9
q_{13} = 0.993 : 1.414, and q_{23} = 0.993 : 1.442. The model, according to the right-hand rule is rotated 45° counterclockwise around the n_3 axis and subsequently 45° counterclockwise around the n_1 axis. We perform wave extrapolation using the exact dispersion relation and compare it with the results from the proposed approximation (equation 45), as well as the weak-anisotropy approximation (Tsvankin, 1997), and the acoustic approximation (Alkhalifah, 2003; Song and Alkhalifah, 2013). The error plots shown in Figure 13 demonstrate noticeably smaller phase errors from the proposed approximation.

![Figure 12: Wavefield extrapolation results in an example of the tilted orthorhombic model. a) Wavefield extrapolation using the exact phase velocity. b) Wavefield extrapolation using the proposed phase-velocity approximation (equation 45). The wavefields are virtually identical.](image)

**DISCUSSION**

Our choice of Muir-Dellinger parametrization leads naturally to a four-parameter velocity approximation in TI media and a nine-parameter approximation in orthorhombic media. The approximations are improved by shifted-hyperboloid functional form. Although highly accurate, these approximations require the same number of parameters as the exact expressions. The benefits of their introduction may not be apparent in the case of phase velocity but are apparent in the consideration of group-velocity approximations because the exact expressions for group velocity in both types of media can be prohibitively complex and cannot be expressed easily in terms of group angle. As observed by previous researchers, the sufficient number of parameters to describe qP wave propagation in TI and orthorhombic media is smaller: three and six respectively. Therefore, we apply the novel relationships between anisotropic parameters summarized in Figure 1 to effectively reduce the number of parameters from the proposed four- and nine-parameter approximations to three- and six-parameter approximations respectively.
Figure 13: Errors in wavefield extrapolation results in an example of the tilted orthorhombic. a) Absolute error of the weak-anisotropy phase-velocity approximation. b) Absolute error of the acoustic phase-velocity approximation. c) Absolute error of the proposed phase-velocity approximation (six-parameter). d) Absolute error of the proposed phase-velocity approximation (nine-parameter).
Apart from the functional form proposed in this paper, many other forms of phase-velocity and group-velocity approximations, especially in TI media, have been extensively investigated in the past (e.g. Alkhalifah and Tsvankin, 1995; Tsvankin, 1996; Mensch and Rasolofosaon, 1997; Pšenčík and Gajewski, 1998; Alkhalifah, 1998, 2000a,b; Farra, 2001; Stopin, 2001; Zhang and Uren, 2001; Farra and Pšenčík, 2003; Daley et al., 2004; Ursin and Stovas, 2006; Fomel and Stovas, 2010; Stovas, 2010; Aleixo and Schleicher, 2010; Farra and Pšenčík, 2013; Hao and Stovas, 2014). While some of them are based on physical assumptions, others are derived purely from mathematical arguments. Our proposed approximation is an alternative, which provides both accuracy and connection with the physical wave phenomena. They are based on the original Muir-Dellinger approximations (Muir and Dellinger, 1985; Dellinger et al., 1993), which were derived on the basis of perturbation from elliptical phase-velocity surfaces. The primary advantage of the Muir-Dellinger parameterization is the ease of conversion between the phase- and group-velocity approximations (e.g. equations 23 and 24), which provides practical convenience. Alternative highly accurate form for phase- and group-velocity approximations is the generalized moveout approximation (Fomel and Stovas, 2010), which was recently applied to anisotropic velocity approximations by Hao and Stovas (2014) and Sripanich and Fomel (2015).

Our approximations are readily applicable to approximate phase and group velocities in the case of transversely isotropic and orthorhombic media whose symmetry axis is aligned with the coordinate axis, e.g., VTI, HTI, and VOR. In the case of TTI (tilted tranversly isotropic) and TOR (tilted orthorhombic), the coordinates simply need to be rotated via Bond transformation before applying the proposed approximations.

**CONCLUSIONS**

We have introduced novel forms of anelliptic approximations for qP velocities in TI and orthorhombic media. The first modification is an empirical connection between $q_1$ and $q_3$ parameters, which depends on the dominant lithology. The second modification is a new functional form of the phase- and group-velocity approximations, allowing up to fourth-order fitting along both symmetry and non-symmetry axes. As a result of these modifications, we arrive at highly accurate four-parameter approximations and new three-parameter approximations for TI media with better accuracy than previously suggested three-parameter approximations for both phase and group velocities. On the basis of the modified anelliptic approximations in TI media, we also propose anelliptic approximations for qP velocities in orthorhombic media, which can be implemented using either six or nine parameters. The proposed orthorhombic phase-velocity approximation maintains the algebraic symmetry and appears to be a more accurate alternative to previously proposed approximations. The group-velocity approximation has an analogous functional form and is also very accurate. The superior accuracy of the proposed phase-velocity approximations is confirmed additionally using wave extrapolation experiments.
ACKNOWLEDGMENTS

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APPENDIX A: UNCERTAINTY ANALYSIS

In this appendix, we study the variation of the phase-velocity expression in TI media with respect to different choices of parameters, particularly Thomsen’s parameters and the proposed Muir-Dellinger parameters. To study resolution, we use the general formula,

\[ R_{ij} = \int_{0}^{\pi/2} \frac{\partial V^2}{\partial m_i} \frac{\partial V^2}{\partial m_j} d\theta, \]

(A-1)

where \( V \) is the exact phase-velocity expression (equation 9), \( m_i \) and \( m_j \) are two of the four parameters present in the expression, and \( \theta \) is the phase angle measured from vertical. The matrix \( R_{ij} \), in both cases, is computed based on the stiffness tensor of Greenhorn shales given in Table 6. The results are shown in Tables 10 and 11. Note that the matrix is symmetric, so the values are shown only on one side of the diagonal.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( V_{P0} )</th>
<th>( V_{S0} )</th>
<th>( \epsilon )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_{P0} )</td>
<td>87.11</td>
<td>0.467</td>
<td>105.39</td>
<td>25.19</td>
</tr>
<tr>
<td>( V_{S0} )</td>
<td></td>
<td>0.005</td>
<td>0.649</td>
<td>0.20</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td></td>
<td></td>
<td>181.74</td>
<td>22.54</td>
</tr>
<tr>
<td>( \delta )</td>
<td></td>
<td></td>
<td></td>
<td>13.16</td>
</tr>
</tbody>
</table>

Table 10: \( R_{ij} \) of the exact phase-velocity expression with the two considered Thomsen parameters are denoted in each row and column.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( w_1 )</th>
<th>( w_3 )</th>
<th>( q_1 )</th>
<th>( q_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>0.576</td>
<td>0.143</td>
<td>0.651</td>
<td>0.599</td>
</tr>
<tr>
<td>( w_3 )</td>
<td></td>
<td>0.538</td>
<td>0.257</td>
<td>1.061</td>
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<tr>
<td>( q_1 )</td>
<td></td>
<td></td>
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<td>1.279</td>
</tr>
<tr>
<td>( q_3 )</td>
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<td></td>
<td></td>
<td>4.124</td>
</tr>
</tbody>
</table>

Table 11: \( R_{ij} \) of the exact phase-velocity expression with the two considered anelliptic parameters are denoted in each row and column.

Table 10 shows a significantly larger correlation between the change in phase velocity with \( V_{P0} \) in comparison with that of \( V_{S0} \), which agrees with the general assumption.
of the independency of \( V_{S0} \) in qP velocities approximations. Likewise, the effect from \( \epsilon \) has a higher correlation with the change of phase velocity than \( \delta \) because the exact qP phase-velocity formula (equation 9) can be expressed in terms of Thomsen parameters with \( \epsilon \) corresponding to the lower order of \( \sin \theta \) than \( \delta \). Moreover, \( \epsilon \) and \( \delta \) also have high correlation with \( V_{P0} \), which is apparent from their definitions.

Table 11 shows relatively similar correlations from \( w_1 \) and \( w_3 \) to the change in exact phase velocity suggesting a more symmetric contribution from both parameters. The dimensionless anelliptic parameters \( q_1 \) and \( q_3 \) exhibit a strong correlation, which is consistent with the relationships shown in Figure 1.

By ignoring the effect of \( V_{S0} \) in the case of Thomsen parameters or using the relationship between \( q_1 \) and \( q_3 \) (Figure 1) to reduce the number of parameters to three, we can transform the matrix \( R_{ij} \) from \( 4 \times 4 \) to \( 3 \times 3 \) (\( \tilde{R}_{ij} \)). Note that the matrix for Thomsen parameters is similar to Table 10 with the omittance of the row and column associated with \( V_{S0} \). Table 12 shows the three-parameter matrix for anelliptic parameters with similar behavior of relatively equal correlations from \( w_1 \) and \( w_3 \) as before.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( w_1 )</th>
<th>( w_3 )</th>
<th>( q_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>0.578</td>
<td>0.144</td>
<td>1.166</td>
</tr>
<tr>
<td>( w_3 )</td>
<td></td>
<td>0.534</td>
<td>1.286</td>
</tr>
<tr>
<td>( q_1 )</td>
<td></td>
<td></td>
<td>7.411</td>
</tr>
</tbody>
</table>

Table 12: \( \tilde{R}_{ij} \) of the exact phase-velocity expression with the two considered anelliptic parameters are denoted in each row and column.

To better visualize the variational effect from the change in the three parameters in both cases, we follow the approach of Osypov et al. (2008), compute the quadratic form of \( \tilde{R}_{ij} \) and plot its contour at a given amount of change in the exact phase velocity expression,

\[
\Delta V^2 = x^T \tilde{R}_{ij} x ,
\]

where \( x \) denotes the vector of parameter variations: \( [\Delta V_{P0}, \Delta \epsilon, \Delta \delta]^T \) or \( [\Delta w_1, \Delta w_3, \Delta q_3]^T \) and \( \tilde{R}_{ij} \) is computed at the known values of the anisotropic parameters of the model (Greenhorn shales). The resultant plots are shown in Figure A-1. For Thomsen’s parameters, Figure A-1a shows a strongly oblate ellipsoid with high degree of deviation (stretch) from a sphere for all three parameters. On the contrary, Figure A-1b shows oblate ellipsoid with smaller deviation suggesting that the Muir-Dellinger parameters may represent a more orthogonal parameterization scheme than Thomsen’s parameters. This observation is important for the problem of estimating anisotropic parameters, which goes beyond the scope of this paper.

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Figure A-1: Ellipsoids obtained from the quadratic form of $\tilde{R}_{ij}$ in the case of (a) Thomsen parameters b) anelliptic parameters.

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