On anelliptic approximations for qP velocities in transversally isotropic media[a]

[a]Published in Geophysical Prospecting, v. 52, 247-259 (2004)

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ABSTRACT

I develop a unified approach for approximating phase and group velocities of qP seismic waves in a transversally isotropic medium with the vertical axis of symmetry (VTI). While the exact phase velocity expressions involve four independent parameters to characterize the elastic medium, the proposed approximate expressions use only three parameters. This makes them more convenient for use in surface seismic experiments, where estimation of all the four parameters is problematic. The three-parameter phase-velocity approximation coincides with the previously published "acoustic" approximation of Alkhalifah. The group velocity approximation is 'new and noticeably more accurate than some of the previously published approximations. I demonstrate an application of the group velocity approximation for finite-difference computation of traveltimes.

INTRODUCTION

Anellipticity (deviation from ellipse) is an important characteristic of elastic wave propagation. One of the simplest and yet practically important cases of anellipticity occurs in transversally isotropic media with the vertical axis of symmetry (VTI). In this type of media, the phase velocities of qSH waves and the corresponding wavefronts are elliptic, while the phase and group velocities of qP and qSV waves may exhibit strong anellipticity (Tsvankin, 2001).

The exact expressions for the phase velocities of qP and qSV waves in VTI media involve four independent parameters. However, it has been observed that only three parameters influence wave propagation and are of interest to surface seismic methods (Alkhalifah and Tsvankin, 1995). Moreover, the exact expressions for the group velocities in terms of the group angle are difficult to obtain and too cumbersome for practical use. This explains the need for developing practical three-parameter approximations for both group and phase velocities in VTI media.

Numerous different successful approximations have been previously developed (Byun et al., 1989; Dellinger et al., 1993; Alkhalifah and Tsvankin, 1995; Alkhalifah, 1998, 2000b; Schoenberg and de Hoop, 2000; Stopin, 2001; Zhang and Uren,

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2001). In this paper, I attempt to construct a unified approach for deriving anelliptic approximations.

The starting point is the anelliptic approximation of Muir (Muir and Dellinger, 1985; Dellinger et al., 1993). Although not the most accurate for immediate practical use, this approximation possesses remarkable theoretical properties. The Muir approximation correctly captures the linear part of anelliptic behavior. It can be applied to find more accurate approximations with nonlinear dependence on the anelliptic parameter. A particular way of "unlinearizing" the linear approximation is the shifted hyperbola approach, familiar from the isotropic approximations in vertically inhomogeneous media (Malovichko, 1978; Sword, 1987; de Bazelaire, 1988; Castle, 1994) and from the theory of Stolt stretch (Stolt, 1978; Fomel and Vaillant, 2001). I show that applying this idea to approximate the phase velocity of qP waves leads to the known "acoustic" approximation of Alkhalifah (1998, 2000a), derived in a different way. Applying the same approach to approximate the group velocity of qP waves leads to a new remarkably accurate three-parameter approximation.

One practical use for the group velocity approximation is traveltime computations, required for Kirchhoff imaging and tomography. In the last part of the paper, I show examples of finite-difference traveltime computations utilizing the new approximation.

EXACT EXPRESSIONS

Wavefront propagation in the general anisotropic media can be described with the anisotropic eikonal equation

$$v^2\left(\frac{\nabla T}{|\nabla T|},\mathbf{x}\right)\,|\nabla T|^2 = 1\;, \tag{1}$$

where $\mathbf{x}$ is a point in space, $T(\mathbf{x})$ is the traveltime at that point for a given source, and $v(\mathbf{n},\mathbf{x})$ is the *phase velocity* in the phase direction $\mathbf{n} = \frac{\nabla T}{|\nabla T|}$.

In the case of VTI media, the three modes of elastic wave propagation (qSH, qSV, and qP) have the following well-known explicit expressions for the phase velocities (Gassmann, 1964):

$$v_{SH}^2(\mathbf{n},\mathbf{x}) = m\,\sin^2\theta + l\,\cos^2\theta\;; \tag{2}$$

$$v_{SV}^2(\mathbf{n},\mathbf{x}) = \frac{1}{2}\left[(a+l)\,\sin^2\theta + (c+l)\,\cos^2\theta\right] - \frac{1}{2}\sqrt{\left[(a-l)\,\sin^2\theta - (c-l)\,\cos^2\theta\right]^2 + 4\,(f+l)^2\,\sin^2\theta\,\cos^2\theta}\;; \tag{3}$$

$$v_{P}^2(\mathbf{n},\mathbf{x}) = \frac{1}{2}\left[(a+l)\,\sin^2\theta + (c+l)\,\cos^2\theta\right] + \frac{1}{2}\sqrt{\left[(a-l)\,\sin^2\theta - (c-l)\,\cos^2\theta\right]^2 + 4\,(f+l)^2\,\sin^2\theta\,\cos^2\theta}\;, \tag{4}$$

where, in the notation of Backus (1962) and Berryman (1979), $a = c_{11}$, $c = c_{33}$, $f = c_{13}$, $l = c_{55}$, $m = c_{66}$, $c_{ij}(\mathbf{x})$ are the density-normalized components of the elastic tensor, and θ is the phase angle between the phase direction $\mathbf{n}$ and the axis of symmetry.

The group velocity describes the propagation of individual ray trajectories $\mathbf{x}(\tau)$. It can be determined from the phase velocity using the general expression

$$\mathbf{V} = \frac{d\mathbf{x}}{d\tau} = v\mathbf{n} + \left(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T\right) \nabla_{\mathbf{n}} v \;, \tag{5}$$

where $\mathbf{I}$ denotes the identity matrix, $\mathbf{n}^T$ stands for the transpose of $\mathbf{n}$, and $\nabla_{\mathbf{n}} v$ is the gradient of v with respect to $\mathbf{n}$. The two terms in equation (5) are clearly orthogonal to each other. Therefore, the group velocity magnitude is (Postma, 1955; Berryman, 1979; Byun, 1984)

$$V = |\mathbf{V}| = \sqrt{v^2 + v_\theta^2} \;, \tag{6}$$

where

$$v_\theta^2 = \left|\left(\mathbf{I} - \mathbf{n}\,\mathbf{n}^T\right) \nabla_{\mathbf{n}} v\right|^2 = \left|\nabla_{\mathbf{n}} v\right|^2 - \left|\mathbf{n} \cdot \nabla_{\mathbf{n}} v\right|^2 \;. \tag{7}$$

The group velocity has a particularly simple form in the case of elliptic anisotropy. Specifically, the phase velocity squared has the quadratic form

$$v_{\text{ell}}^2(\mathbf{n},\mathbf{x}) = \mathbf{n}^T\,\mathbf{A}(\mathbf{x})\,\mathbf{n} \tag{8}$$

with a symmetric positive-definite matrix $\mathbf{A}$, and the group velocity is

$$\mathbf{V}_{\text{ell}} = \mathbf{A}\,\mathbf{p} \;, \tag{9}$$

where $\mathbf{p} = \nabla T = \mathbf{n}/v(\mathbf{n},\mathbf{x})$. The corresponding group slowness squared has the explicit expression

$$\frac{1}{V_{\text{ell}}^2(\mathbf{N},\mathbf{x})} = \mathbf{N}^T\,\mathbf{A}^{-1}(\mathbf{x})\,\mathbf{N} \;, \tag{10}$$

where $\mathbf{N}$ is the group direction, and $\mathbf{A}^{-1}$ is the matrix inverse of $\mathbf{A}$. For example, the elliptic expression (2) for the phase velocity of qSH waves in VTI media transforms into a completely analogous expression for the group slowness

$$\frac{1}{V_{SH}^2(\mathbf{N},\mathbf{x})} = M\,\sin^2\Theta + L\,\cos^2\Theta \tag{11}$$

where $M = 1/m$, $L = 1/l$, and Θ is the angle between the group direction $\mathbf{N}$ and the axis of symmetry.

The situation is more complicated in the anelliptic case. Figure 1 shows the qP and qSV phase velocity profiles in a transversely isotropic material – Greenhorn shale (Jones and Wang, 1981), which has the parameters $a = 14.47\,\text{km}^2/\text{s}^2$, $l = 2.28\,\text{km}^2/\text{s}^2$, $c = 9.57\,\text{km}^2/\text{s}^2$, and $f = 4.51\,\text{km}^2/\text{s}^2$. Figure 2 shows the corresponding group velocity profiles. The non-convexity of the qSV phase velocity causes a multi-valued (triplicated) group velocity profile. The shapes of all the surfaces are clearly anelliptic.

A simple model of anellipticity is suggested by the Muir approximation (Muir and Dellinger, 1985; Dellinger et al., 1993), reviewed in the next section.

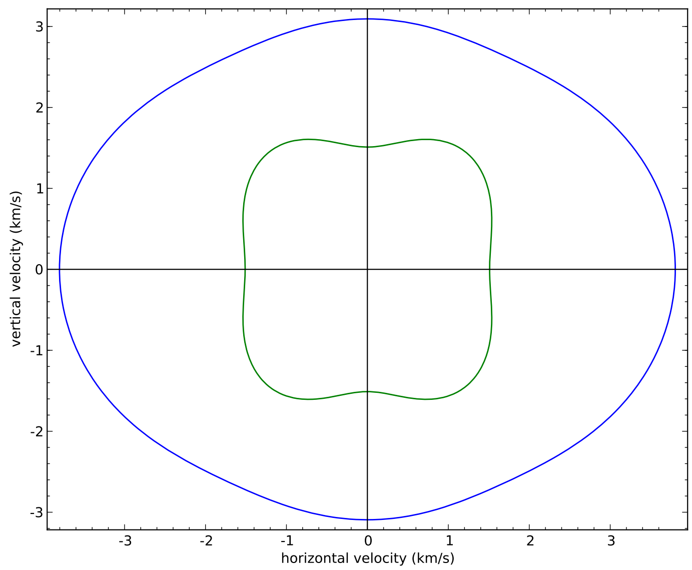


Figure 1: Phase velocity profiles for qP (outer curve) and qSV (inner curve) waves in a transversely isotropic material (Greenhorn shale).

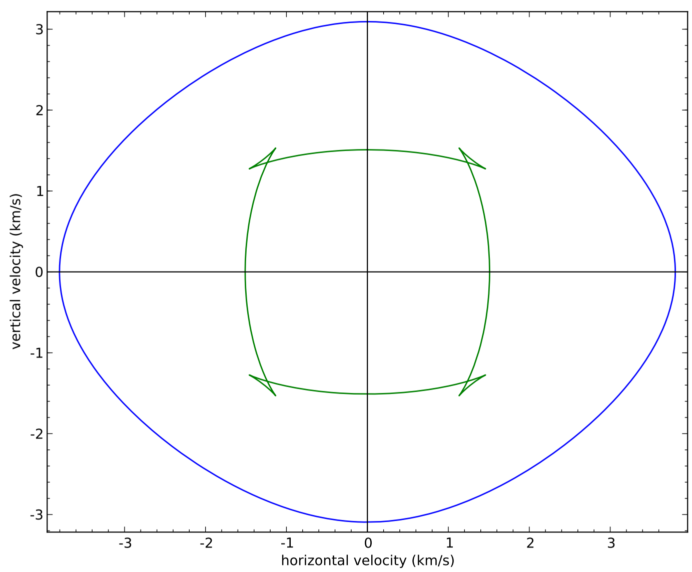


Figure 2: Group velocity profiles for qP (outer curve) and qSV (inner curve) waves in a transversely isotropic material (Greenhorn shale).

MUIR APPROXIMATION

Muir and Dellinger (1985) suggested representing anelliptic qP phase velocities with the following approximation:

$$v_P^2(\theta) \approx e(\theta) + \frac{(q-1)\,a\,c\,\sin^2\theta\,\cos^2\theta}{e(\theta)}\;, \tag{12}$$

where $e(\theta)$ is the elliptical part of the velocity, defined by

$$e(\theta) = a\,\sin^2\theta + c\,\cos^2\theta\;, \tag{13}$$

and q is the anellipticity coefficient ($q = 1$ in case of elliptic velocities). Approximation (12) uses only three parameters to characterize the medium (a, c, and q) as opposed to the four parameters (a, c, l, and f) in the exact expression.

There is some freedom in choosing an appropriate value for the coefficient q. Assuming near-vertical wave propagation and the vertical axis of symmetry (a VTI medium) and fitting the curvature ($d^2 v_P/d\theta^2$) of the exact phase velocity (4) near the vertical phase angle ($\theta = 0$), leads to the definition (Dellinger et al., 1993)

$$q = \frac{l\,(c-l) + (l+f)^2}{a\,(c-l)}\;. \tag{14}$$

In terms of Thomsen's elastic parameters ϵ and δ (Thomsen, 1986) and the elastic parameter η of Alkhalifah and Tsvankin (1995),

$$q = \frac{1+2\,\delta}{1+2\epsilon} = \frac{1}{1+2\,\eta}\;. \tag{15}$$

This confirms the direct relationship between η and anellipticity. If we were to fit the phase velocity curvature near the horizontal axis $\theta = \pi/2$ (perpendicular to the axis of symmetry), the appropriate value for q would be

$$\hat{q} = \frac{l\,(a-l) + (l+f)^2}{c\,(a-l)}\;. \tag{16}$$

Muir and Dellinger (1985) also suggested approximating the VTI group velocity with an analogous expression

$$\frac{1}{V_P^2(\Theta)} \approx E(\Theta) + \frac{(Q-1)\,A\,C\,\sin^2\Theta\,\cos^2\Theta}{E(\Theta)} \tag{17}$$

where $A = 1/a$, $C = 1/c$, $Q = 1/q$, Θ is the group angle, and $E(\Theta)$ is the elliptical part:

$$E(\Theta) = A\,\sin^2\Theta + C\,\cos^2\Theta\;. \tag{18}$$

Equations (12) and (17) are consistent in the sense that both of them are exact for elliptic anisotropy ($q = Q = 1$) and accurate to the first order in $(q-1)$ or $(Q-1)$ in the general case of transversally isotropic media.

To the same approximation order, the connection between the phase and group directions is

$$\tan\Theta = \tan\theta\,\frac{a}{c}\,\left(1-(q-1)\,\frac{a\,\sin^2\theta - c\,\cos^2\theta}{a\,\sin^2\theta + c\,\cos^2\theta}\right)\;. \tag{19}$$

SHIFTED HYPERBOLA APPROXIMATION FOR THE PHASE VELOCITY

Despite the beautiful symmetry of Muir's approximations (12) and (17), they are less accurate in practice than some other approximations, most notably the weak anisotropy approximation of Thomsen (1986), which can be written as (Tsvankin, 1996)

$$v_P^2(\theta) \approx c\,\left(1+2\,\epsilon\,\sin^4\theta + 2\,\delta\,\sin^2\theta\,\cos^2\theta\right)\;, \tag{20}$$

where

$$\epsilon = \frac{a-c}{2\,c} \quad \text{and} \quad \delta = \frac{(l+f)^2-(c-l)^2}{2\,c\,(c-l)}\;. \tag{21}$$

Note that both approximations involve the anellipticity factor ($q-1$ or $\epsilon-\delta$) in a linear fashion. If the anellipticity effect is significant, the accuracy of Muir's equations can be improved by replacing the linear approximation with a nonlinear one. There are, of course, infinitely many nonlinear expressions that share the same linearization. In this study, I focus on the shifted hyperbola approximation, which follows from the fact that an expression of the form

$$x+\frac{\alpha}{x} \tag{22}$$

is the linearization (Taylor series expansion) of the form

$$x\,(1-s)+s\,\sqrt{x^2+\frac{2\,\alpha}{s}} \tag{23}$$

for small α. Linearization does not depend on the parameter s, which affects only higher-order terms in the Taylor expansion. Expression (23) is reminiscent of the shifted hyperbola approximation for normal moveout in vertically heterogeneous media (Malovichko, 1978; Sword, 1987; de Bazelaire, 1988; Castle, 1994) and the Stolt stretch correction in the frequency-wavenumber migration (Stolt, 1978; Fomel and Vaillant, 2001). It is evident that Muir's approximation (12) has exactly the right form (22) to be converted to the shifted hyperbola approximation (23).

Thus, we seek an approximation of the form

$$v_P^2(\theta) \approx e(\theta)\,(1-s)+s\,\sqrt{e^2(\theta)+\frac{2\,(q-1)\,a\,c\,\sin^2\theta\,\cos^2\theta}{s}} \tag{24}$$

with $e(\theta)$ defined by equation (13). The plan is to select a value of the additional parameter s to fit the exact phase velocity expression (4) and then to constrain

s so that it depends only on the three parameters already present in the original approximation (12).

One can verify that the velocity curvature $d^2v_P/d\theta^2$ around the vertical axis $\theta = 0$ for approximation (24) depends on the chosen value of q but does not depend on the value of the shift parameter s. This means that the velocity profile $v_P(\theta)$ becomes sensitive to s only further away from the vertical direction. This separation of influence between the approximation parameters is an important and attractive property of the shifted hyperbola approximation. I find an appropriate value for s by fitting additionally the fourth-order derivative $d^4v_P/d\theta^4$ at $\theta = 0$ to the corresponding derivative of the exact expression. The fit is achieved when s has the value

$$s = \frac{c-l}{2}\,\frac{(a-l)\,(c-l)-(l+f)^2}{a\,(c-l)^2-c\,(l+f)^2}\;. \tag{25}$$

It is more instructive to express it in the form

$$s = \frac{1}{2}\,\frac{(a-c)\,(q-1)\,(\hat{q}-1)}{a\,\left(1-\hat{q}-q\,(1-q)\right)-c\,\left((\hat{q}-1)^2+\hat{q}\,(q-\hat{q})\right)}\;, \tag{26}$$

where q and $\hat{q}$ are defined by equations (14) and (16). In this form of the expression, $\hat{q}$ appears as the extra parameter that we need to eliminate. This parameter was defined by fitting the velocity profile curvature around the horizontal axis, which would correspond to infinitely large offsets in a surface seismic experiment. One possible way to constrain it is to set $\hat{q}$ equal to q, which implies that the velocity profile has similar behavior near the vertical and the horizontal axes. Setting $\hat{q} \approx q$ in equation (26) yields

$$s \approx \lim_{\hat{q}\to q} s = \frac{1}{2}\;. \tag{27}$$

Substituting (27) in equation (24) produces the final approximation

$$v_P^2(\theta) \approx \frac{1}{2}\,e(\theta) + \frac{1}{2}\,\sqrt{e^2(\theta) + 4\,(q-1)\,a\,c\,\sin^2\theta\,\cos^2\theta}\;. \tag{28}$$

Approximation (28) is exactly equivalent to the *acoustic approximation* of Alkhalifah (1998, 2000a), derived with a different set of parameters by formally setting the S-wave velocity ($l = v_S^2$) in equation (4) to zero. A similar approximation is analyzed by Stopin (2001). Approximation (28) was proved to possess a remarkable accuracy even for large phase angles and significant amounts of anisotropy. Figure 3 compares the accuracy of different approximations using the parameters of the Greenhorn shale. The acoustic approximation appears especially accurate for phase angles up to about 25 degrees and does not exceed the relative error of 0.3% even for larger angles.

SHIFTED HYPERBOLA APPROXIMATION FOR THE GROUP VELOCITY

Similar strategy is applicable for approximating the group velocity. Applying the shifted hyperbola approach to "unlinearize" Muir's approximation (17), we seek an

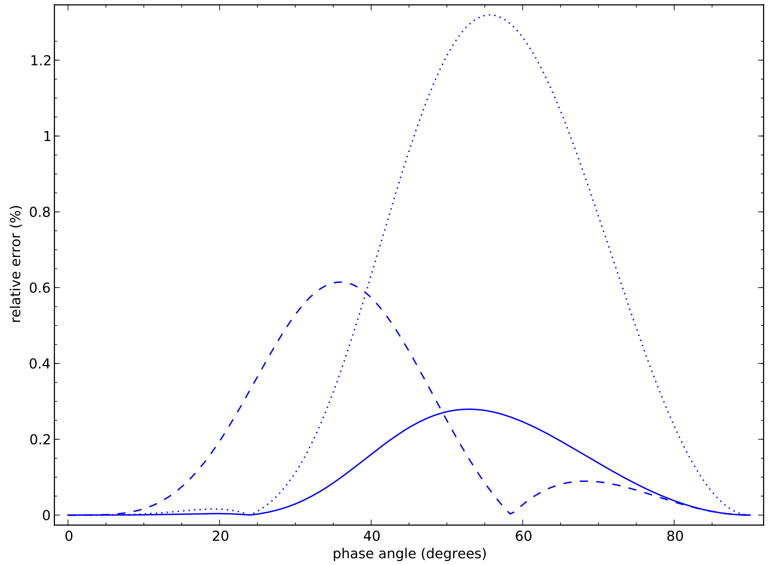


Figure 3: Relative error of different phase velocity approximations for the Greenhorn shale anisotropy. Short dash: Thomsen's weak anisotropy approximation. Long dash: Muir's approximation. Solid line: suggested approximation (similar to Alkhalifah's acoustic approximation.)

approximation of the form

$$\frac{1}{V_P^2(\Theta)} \approx E(\Theta)\,(1-S) + S\,\sqrt{E^2(\Theta) + \frac{2\,(Q-1)\,A\,C\,\sin^2\Theta\,\cos^2\Theta}{S}} \tag{29}$$

An approximation of this form with S set to $1/2$ was proposed earlier by Zhang and Uren (2001). Similarly to the case of the phase velocity approximation, I constrain the value of S by Taylor fitting of the velocity profiles near the vertical angle.

Although there is no simple explicit expression for the transversally isotropic group velocity, we can differentiate the parametric representations of V_P and Θ in terms of the phase angle θ that follow from equation (5). The group velocity is an even function of the angle Θ because of the VTI symmetry. Therefore, the odd-order derivatives are zero at the axis of symmetry ($\Theta = \theta = 0$). Fitting the second-order derivative $d^2V_P/d\Theta^2$ at $\theta = 0$ produces $Q = 1/q = 1 + 2\,\eta$, consistent with Muir's approximation (17). Fitting additionally the fourth-order derivative $d^4V_P/d\Theta^4$ at $\theta = 0$ produces

$$S = \frac{1}{2}\,\frac{\left[(l+f)^2 + l\,(c-l)\right]^2\,\left[(c-l)\,(a-l) - (l+f)^2\right]}{a^2\,c\,(c-l)\,(l+f)^2 - \left[l\,(c-l) + (l+f)^2\right]^3} \tag{30}$$

or, equivalently,

$$S = \frac{1}{2}\,\frac{(C-A)\,(Q-1)\,(\hat{Q}-1)}{C\,\left(\hat{Q}\,(Q^2-Q-1)+1\right) + A\,\left(\hat{Q} - Q^3 + Q^2 - 1\right)}\;, \tag{31}$$

where $\hat{Q} = 1/\hat{q}$. As in the previous section, I approximate the optimal value of S by setting $\hat{Q}$ equal to Q, as follows:

$$S \approx \lim_{\hat{Q} \to Q} S = \frac{1}{2\,(1+Q)} = \frac{1}{4\,(1+\eta)}\;. \tag{32}$$

Selected in this way, the value of S depends on the anelliptic parameter Q (or η) and, for small anellipticity, is close to 1/4, which is different from the value of 1/2 in the approximation of Zhang and Uren (2001).

The final group velocity approximation takes the form

$$\frac{1}{V_P^2(\Theta)} \approx \frac{1+2\,Q}{2\,(1+Q)}\,E(\Theta) + \frac{1}{2\,(1+Q)}\,\sqrt{E^2(\Theta) + 4\,(Q^2-1)\,A\,C\,\sin^2\Theta\,\cos^2\Theta}\;. \tag{33}$$

In Figure 4, the accuracy of approximation (33) is compared with the accuracy of Muir's approximation (17) and the accuracy of the weak anisotropy approximation (Thomsen, 1986) for the elastic parameters of the Greenhorn shale. The weak anisotropy approximation, used in this comparison, is

$$V_P^2(\Theta) \approx c\,\left(1 + 2\,\epsilon\,\sin^4\Theta + 2\,\delta\,\sin^2\Theta\,\cos^2\Theta\right)\;, \tag{34}$$

where ϵ and δ are Thomsen's parameters, defined in equations (21). A similar form (in a different parameterization) was introduced by Byun et al. (1989).

Approximation (33) turns out to be remarkably accurate for this example. It appears nearly exact for group angles up to 45 degrees from vertical and does not exceed 0.3% relative error even at larger angles. It is compared with two other approximations in Figure 5. These are the Zhang-Uren approximation (Zhang and Uren, 2001) and the Alkhalifah-Tsvankin approximation, which follows directly from the normal moveout equation suggested by Alkhalifah and Tsvankin (1995):

$$t^2(x) \approx t_0^2 + \frac{x^2}{V_n^2} - \frac{2\,\eta\,x^4}{V_n^2\,\left[t_0^2\,V_n^2 + (1+2\,\eta)\,x^2\right]}\;, \tag{35}$$

where $t(x)$ is the moveout curve, t_0 is the vertical traveltime, and $V_n = \sqrt{a/(1+2\,\eta)}$ is the NMO velocity. In a homogeneous medium, equation (35) corresponds to the group velocity approximation

$$\frac{1}{V_P^2(\Theta)} \approx \frac{\cos^2\Theta}{V_z^2} + \frac{\sin^2\Theta}{V_n^2} - \frac{2\,\eta\,\sin^4\Theta}{V_n^2\,\left[\cos^2\Theta\,V_n^2/V_z^2 + (1+2\,\eta)\,\sin^2\Theta\right]}\;, \tag{36}$$

where $V_z = \sqrt{c}$. In the notation of this paper, the Alkhalifah-Tsvankin equation (36) takes the form

$$\frac{1}{V_P^2(\Theta)} \approx E(\Theta) + \frac{(Q-1)\,A\,C\,\sin^2\Theta\,\cos^2\Theta}{E(\Theta) + (Q^2-1)\,A\,\sin^2\Theta} \tag{37}$$

and differs from approximation (17) by the correction term in the denominator. Approximation (33) is noticeably more accurate for this example than any of the other approximations considered here.

Another accurate group velocity approximation was suggested by Alkhalifah (2000b). However, the analytical expression is complicated and inconvenient for practical use. The accuracy of Alkhalifah's approximation for the Greenhorn shale example is depicted in Figure 6.

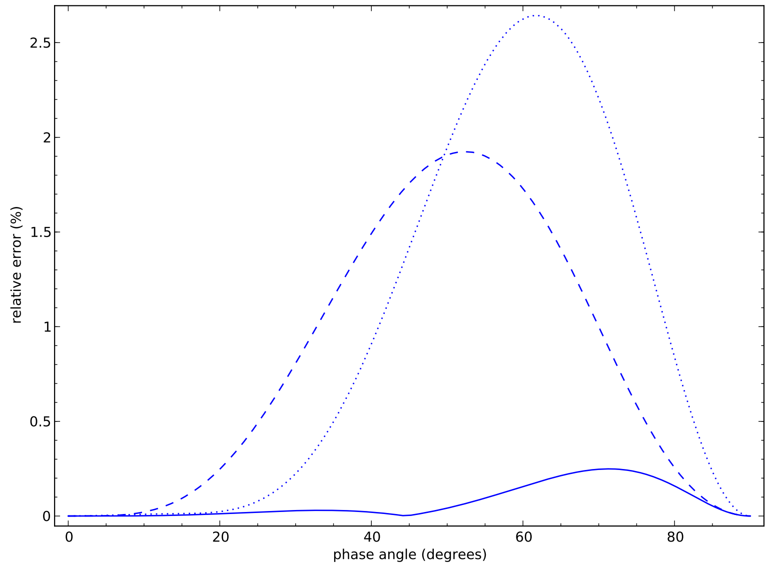


Figure 4: Relative error of different group velocity approximations for the Greenhorn shale anisotropy. Short dash: Thomsen's weak anisotropy approximation. Long dash: Muir's approximation. Solid line: suggested approximation.

It is similarly possible to convert a group velocity approximation into the corresponding moveout equation. In a homogeneous anisotropic medium, the reflection traveltime t as a function of offset x is

$$t(x) = \frac{2\,\sqrt{(x/2)^2 + z^2}}{V_P\left(\arctan\left(\frac{x}{2\,z}\right)\right)}\;, \tag{38}$$

where $z = t_0\,V_P(0)/2$ is the depth of the reflector. The moveout equation corresponding to approximation (33) is

$$\begin{aligned} t^2(x) &\approx \frac{1+2\,Q}{2\,(1+Q)}\,H(x) + \frac{1}{2\,(1+Q)}\,\sqrt{H^2(x) + 4\,(Q^2-1)\,\frac{t_0^2\,x^2}{Q\,V_n^2}} \\ &= \frac{3+4\,\eta}{4\,(1+\eta)}\,H(x) + \frac{1}{4\,(1+\eta)}\,\sqrt{H^2(x) + 16\,\eta\,(1+\eta)\,\frac{t_0^2\,x^2}{(1+2\,\eta)\,V_n^2}}\;, \end{aligned} \tag{39}$$

where $H(x)$ represents the hyperbolic part:

$$H(x) = t_0^2 + \frac{x^2}{Q\,V_n^2} = t_0^2 + \frac{x^2}{(1+2\,\eta)\,V_n^2}\;. \tag{40}$$

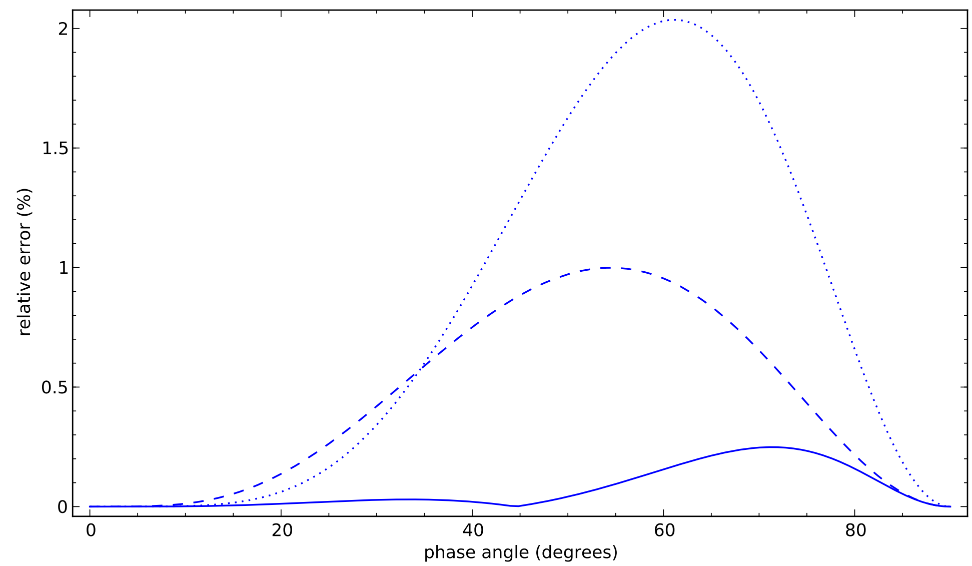


Figure 5: Relative error of different group velocity approximations for the Greenhorn shale anisotropy. Short dash: Alkhalifah-Tsvankin approximation. Long dash: Zhang-Uren approximation. Solid line: suggested approximation.

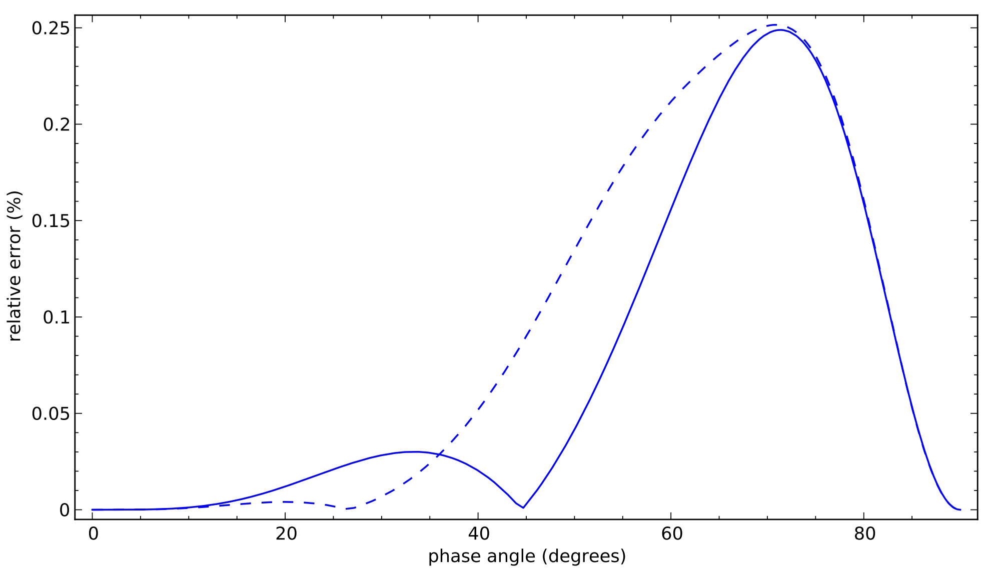


Figure 6: Relative error of different group velocity approximations for the Greenhorn shale anisotropy. Dashed line: Alkhalifah approximation. Solid line: suggested approximation.

For small offsets, the Taylor series expansion of equation (39) is

$$\begin{aligned} t^2(x) &\approx t_0^2 + \frac{x^2}{V_n^2} - (Q-1)\,\frac{x^4}{t_0^2\,V_n^4} + (Q-1)\,(2\,Q^2-1)\,\frac{x^6}{Q\,t_0^4\,V_n^6} + O(x^8) \\ &= t_0^2 + \frac{x^2}{V_n^2} - 2\,\eta\,\frac{x^4}{t_0^2\,V_n^4} + 2\,\eta\,(1+8\,\eta+8\eta^2)\,\frac{x^6}{(1+2\,\eta)\,t_0^4\,V_n^6} + O(x^8)\;. \end{aligned} \quad (41)$$

Figure 7 compares the accuracy of different moveout approximations assuming reflection from the bottom of a homogeneous anisotropic layer of 1 km thickness with the elastic parameters of Greenhorn shale. Approximation (39) appears extremely accurate for half-offsets up to 1 km and does not develop errors greater than 5 ms even at much larger offsets.

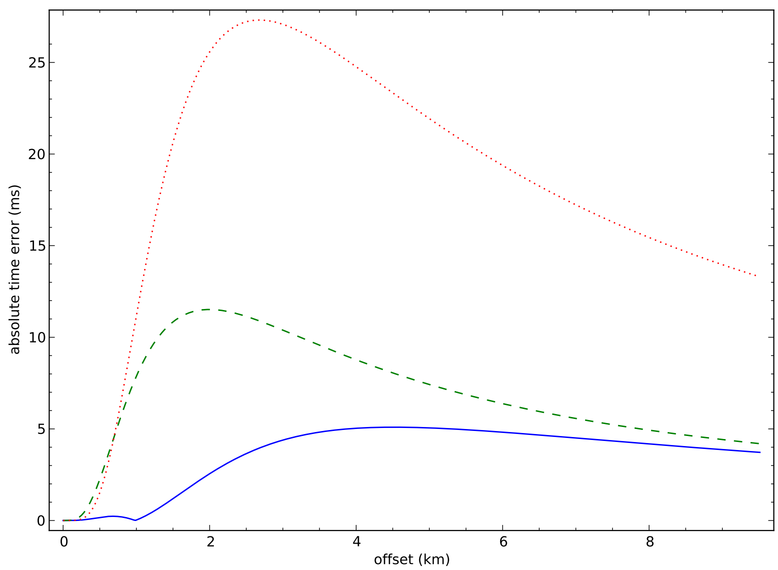


Figure 7: Traveltime moveout error of different group velocity approximations for Greenhorn shale anisotropy. The reflector depth is 1 km. Short dash: Alkhalifah-Tsvankin approximation. Long dash: Zhang-Uren approximation. Solid line: suggested approximation.

It remains to be seen if the suggested approximation proves to be useful for describing normal moveout in layered media. The next section discusses its application for traveltime computation in heterogenous velocity models.

APPLICATION: FINITE-DIFFERENCE TRAVELTIME COMPUTATION

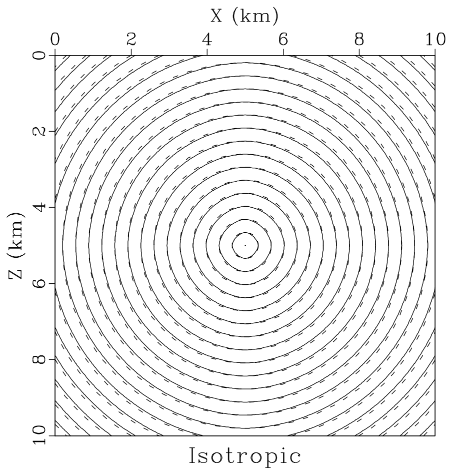
As an essential part of seismic imaging with the Kirchhoff method, traveltime computation has received a lot of attention in the geophysical literature. Finite-difference eikonal solvers (Vidale, 1990; van Trier and Symes, 1991; Podvin and Lecomte, 1991) provide an efficient and convenient way of computing first arrival traveltimes on regular grids. Although they have a limited capacity for imaging complex structures (Geoltrain and Brac, 1993), eikonal solvers can be extended in several different ways to accommodate multiple arrivals (Bevc, 1997; Symes, 1998; Abgrall and Benamou, 1999). A particularly attractive approach to finite-difference traveltime computation is the fast marching method, developed by Sethian (1996) in the general context of level set methods for propagating interfaces (Osher and Sethian, 1988; Sethian, 1999). Sethian and Popovici (1999) adopt the fast marching method for computing seismic isotropic traveltimes. Alternative implementations are discussed by Sun and Fomel (1998), Alkhalifah and Fomel (2001), and Kim (2002). The fast marching method possesses a remarkable numerical stability, which results from a cleverly chosen order of finite-difference evaluation. The order selection scheme resembles expanding wavefronts of Qin et al. (1992) and wavefront tracking of Cao and Greenhalgh (1994).

While the anisotropic eikonal equation (1) operates with phase velocities, the kernel of the fast marching eikonal solver can be interpreted in terms of local ray tracing in a constant-velocity background (Fomel, 1997) and is more conveniently formulated with the help of the group velocity. Sethian and Vladimirsky (2001) present a thorough extention of the fast marching method to anisotropic wavefront propagation in the form of ordered upwind methods. In this paper, I adopt a simplified approach. Anisotropic traveltimes are computed in relation to an isotropic background. At each step of the isotropic fast marching method, the local propagation direction is identified, and the anisotropic traveltimes are computed by local ray tracing with the group velocity corresponding to the same direction. This is analogous to the tomographic linearization approach in ray tracing, where anisotropic traveltimes are computed along ray trajectories, traced in the isotropic background (Chapman and Pratt, 1992). Alkhalifah (2002) and Schneider (2003) present different approaches for linearizing the anisotropic eikonal equation.

Many alternative forms of finite-difference traveltime computation in anisotropic media are presented in the literature (Qin and Schuster, 1993; Dellinger and Symes, 1997; Kim, 1999; Bousquie and Siliqi, 2001; Perez and Bancroft, 2001; Qin and Symes, 2002; Zhang et al., 2002). Although the method of this paper has limited accuracy because of the linearization assumption, it is simple and efficient in practice and serves as an illustration for the advantages of the explicit group velocity approximation (33). For a more accurate and robust extension of the fast marching method for anisotropic traveltime calculation, I recommend the ordered upwind methods of Sethian and Vladimirsky (2001, 2003).

Figure 8 shows finite-difference wavefronts for an isotropic and an anisotropic homogeneous media, compared with the exact solutions. The anisotropic media has the parameters of the Greenhorn shale. The finite-difference error decreases with finer sampling.

Figure 9 shows the first arrival wavefronts (traveltime contours) computed in the anisotropic Marmousi model created by Alkhalifah (1997) in comparison with wavefronts for the isotropic Marmousi model (Versteeg, 1994; TME, 1990). The



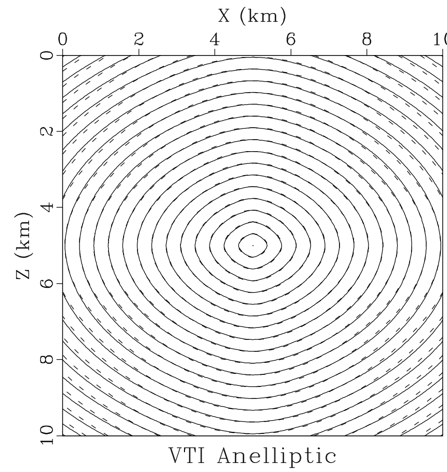


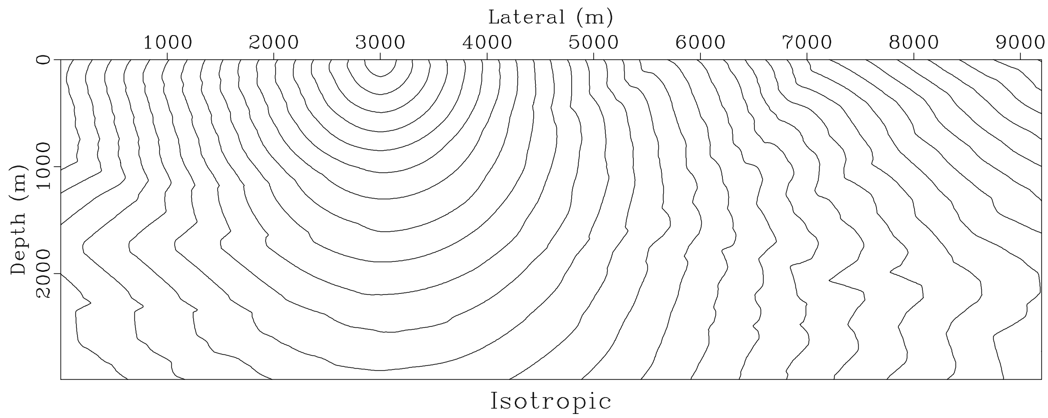
Figure 8: Finite-difference wavefronts in an isotropic (left) and an anisotropic (right) homogeneous media. The anisotropic media has the parameters of the Greenhorn shale. The finite-difference sampling is 100 m. The contour sampling is 0.1 s. Dashed curves indicate the exact solution. The finite-difference error will be reduced at finer sampling.

model parameters are shown in Figure 10. The observed significant difference in the wavefront position suggests a difference in the positioning of seismic images when anisotropy is not properly taken into account.

CONCLUSIONS

I have developed a general approach for approximating both phase and group velocities in a VTI medium. Suggested approximations use three elastic parameters as opposed to the four parameters in the exact phase velocity expression. The phase velocity approximation coincides with the acoustic approximation of Alkhalifah (1998, 2000a) but is derived differently. The group velocity approximation has an analogous form and similar superior approximation properties. It is important to stress that the two approximations do not correspond exactly to each other. The exact group velocity corresponding to the acoustic approximation is different from the approximation derived in this paper and can be too complicated for practical use (Alkhalifah, 1999). The suggested phase and group approximations match each other in the sense that they have analogous approximation accuracy in the dual domains.

The group velocity approximation is useful for approximating normal moveout and diffraction traveltimes in applications to non-hyperbolic velocity analysis and prestack time migration. It is also useful for traveltime computations that require ray tracing in locally homogeneous cells. I have shown examples of such computations utilizing an anisotropic extension of the fast marching finite-difference eikonal solver.



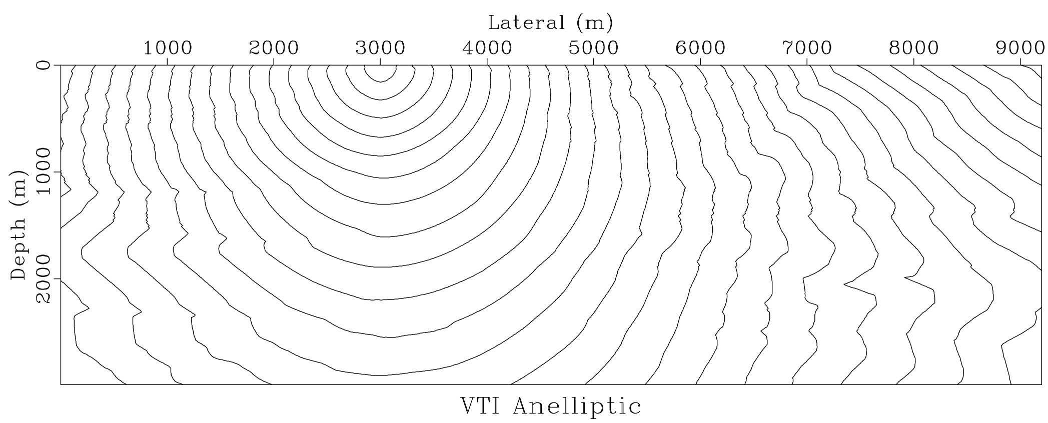


Figure 9: Finite-difference wavefronts in the isotropic (top) and anisotropic (bottom) Marmousi models. A significant shift in the wavefront position suggest possible positioning error when seismic imaging does not take anisotropy into account.

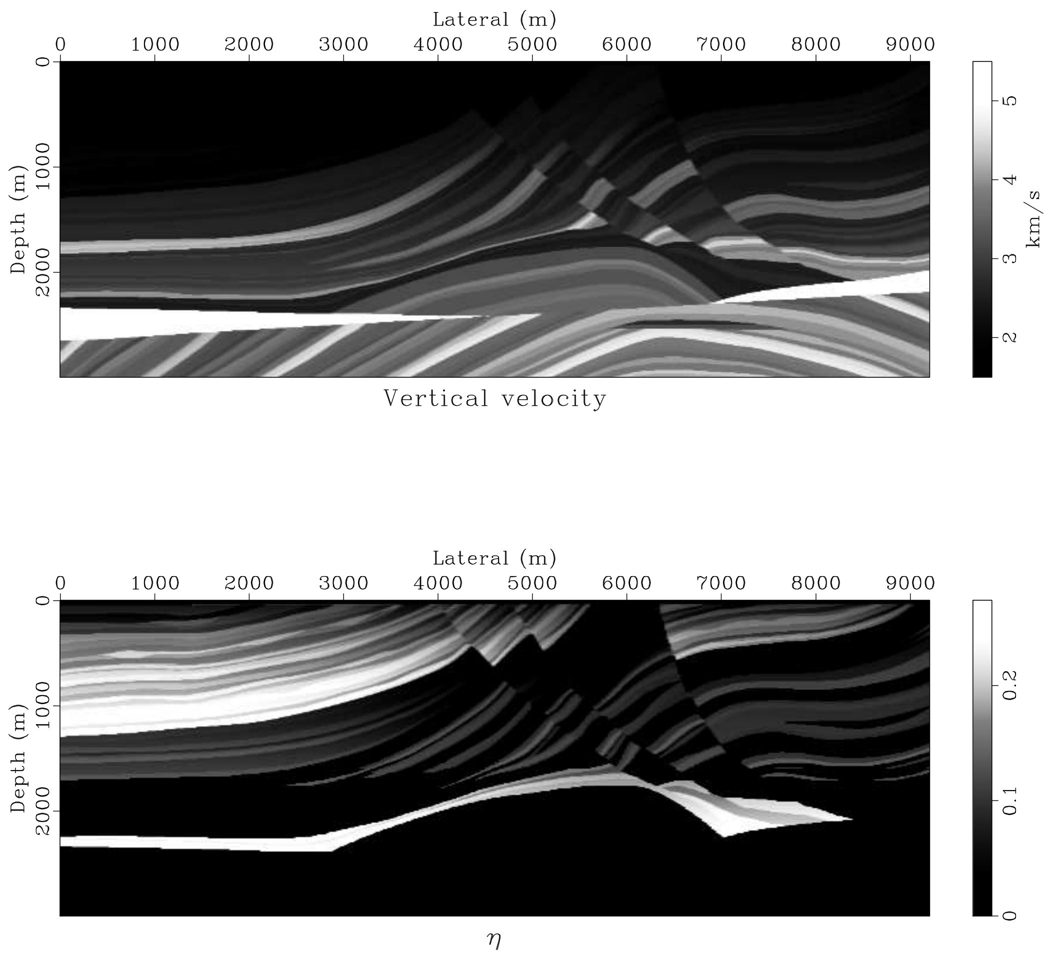


Figure 10: Alkhalifah's anisotropic Marmousi model. Top: vertical velocity. Bottom: anelliptic η parameter. The vertical velocity is taken equal to the NMO velocity V_n.

ACKNOWLEDGMENTS

The paper was improved by helpful suggestions from Tariq Alkhalifah and Paul Fowler.

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